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Measure and Category in Real Analysis

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Contents

Chapter 1

Introduction

1.1 A short general introduction

In this thesis we study three problems of real analysis. The concepts of measure and category play an important role in each chapter.

Geometric measure theory was developed in the early twentieth century, it is concerned with investigating subsets of Euclidean spaces from measure theoretical point of view. The methods are useful in many areas of mathematics, for example in harmonic analysis, probability theory and PDEs. Due to the applications of fractal sets and random processes it became very important not only in mathematics but also in other sciences.

Baire's category theory is less popular, but also very important in mathematics. It is a basic tool in real analysis, classical functional analysis, ergodic theory and harmonic analysis. Several theorem in measure theory has a category analogue and vice versa, this thesis deals with this duality, too. A central application of the category theory in real analysis is describing the properties of the generic continuous function.

One of the most important tools of geometric measure theory is dimension. First Hausdorff discovered the so-called Hausdorff dimension, and the box-counting dimension is due to Tricot. In Chapter 2 we introduce a new concept of dimension for metric spaces, the so called *topological Hausdorff dimension*. It is defined by a very natural combination of the definitions of the topological dimension and the Hausdorff dimension. The value of the topological Hausdorff dimension is always between the topological dimension and the Hausdorff dimension, in particular, this new dimension is a non-trivial lower estimate for the Hausdorff dimension.

We examine the basic properties of this new notion of dimension, compare it to other well-known notions, determine its value for some classical fractals such as the Sierpiński carpet, the von Koch snowflake curve, Kakeya sets, the trail of the Brownian motion, etc.

As our first application, we generalize the celebrated result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. They proved that certain curves show up in the limit set when passing a critical probability, and we prove that actually 'thick' families of curves show up, where roughly speaking the word thick means that the curves can be parametrized in a natural way by a set of large Hausdorff dimension. The proof of this is basically a lower estimate of the topological Hausdorff dimension of the limit set. For the sake of completeness, we also give an upper estimate and conclude that in the non-trivial cases the topological Hausdorff dimension is almost surely strictly below the Hausdorff dimension.

Finally, as our second application, we show that the topological Hausdorff dimension is precisely the right notion to describe the Hausdorff dimension of the level sets of the generic continuous function (in the sense of Baire category) defined on a compact metric space. The results of this chapter are joint work with Z. Buczolich and M. Elekes.

Chapter 3 deals with the duality between measure and category. Erdős and Sierpiński proved that consistently exists a duality between meager and null subsets of $\mathbb{R}$, and Ryll-Nardzewski asked whether it can be additive. First Bartoszyński gave a negative answer to the question in the case 2^ω instead of $\mathbb{R}$, then Kysiak answered the original question. We generalize their results for uncountable LCA Polish groups, where nullsets are with respect to the unique Haar measure. It is almost the most general space where the question makes sense, maybe commutativity can be left.

Chapter 4 is based on classical real function theory. We call a function *vertically rigid* if its multiplicated copies are isometric. We characterize the continuous vertically rigid functions of one and two variables, the one variable case solves Janković's conjecture. We answer a question of Cain, Clark and Rose. We give results in the Lebesgue (Baire) measurable cases, where we use methods of geometric measure (Baire category) theory. Besides analysis, we use algebraic methods to solve certain function equations. The results of this chapter are joint work with M. Elekes.

The following sections contain longer and more precise introduction to the questions and results.

1.2 The topological Hausdorff dimension

The term 'fractal' was introduced by Mandelbrot in his celebrated book [27]. He formally defined a subset of a Euclidean space to be a fractal if its topological dimension is strictly smaller than its Hausdorff dimension. This is just one example to illustrate the fundamental role these two notions of dimension play in the study of fractal sets. To mention another such example, let us recall that the topological dimension of a metric space X is the infimum of the Hausdorff dimensions of the metric spaces homeomorphic to X, see [22].

The main goal of this chapter is to introduce a new concept of dimension, the so called *topological Hausdorff dimension*, that interpolates the two above mentioned dimensions in a very natural way. Let us recall the definition of the (small inductive) topological dimension.

Definition 1.1. Set $\dim_t \emptyset = -1$. The *topological dimension* of a non-empty metric space X is defined by induction as

$$\dim_t X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_t \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

Our new dimension will be defined analogously, however, note that this second definition will not be inductive, and also that it can attain non-integer values as well. The Hausdorff dimension of a metric space X is denoted by $\dim_H X$, see e.g. [14] or [29]. In this chapter we adopt the convention that $\dim_H \emptyset = -1$.

Definition 1.2. Set $\dim_{tH} \emptyset = -1$. The *topological Hausdorff dimension* of a non-empty metric space X is defined as

$$\dim_{tH} X = \inf\{d : X \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\}.$$

(Both notions of dimension can attain the value ∞ as well.)

It was not this analogy that initiated the study of this new concept. Our original motivation was that this notion grew out naturally from our investigations of the following topic. B. Kirchheim proved in [24] that for the generic continuous function (in the sense of Baire category) defined on $[0,1]^d$, for every $y \in \operatorname{int} f([0,1]^d)$ we have $\dim_H f^{-1}(y) = d-1$, that is, as one would expect, 'most' level sets are of Hausdorff dimension $d-1$. The next problem is about generalizations of this result to fractal sets in place of $[0,1]^d$.

Problem 1.3. *Describe the Hausdorff dimension of the level sets of the generic continuous function (in the sense of Baire category) defined on a compact metric space.*

It has turned out that the topological Hausdorff dimension is the right concept to deal with this problem. We will essentially prove that the value $d-1$ in Kirchheim's result has to be replaced by $\dim_{tH} K - 1$, see the end of this introduction or Section 2.5 for the details.

We would also like to mention another potentially very interesting motivation of this new concept. Unlike most well-known notions of dimension, such as packing or box-counting dimensions, the topological Hausdorff dimension is smaller than the Hausdorff dimension. As it is often an important and difficult task to estimate the Hausdorff dimension from below, this gives another reason why to study the topological Hausdorff dimension.

It is also worth mentioning that there is another recent approach by M. Urbański [35] to combine the topological dimension and the Hausdorff dimension. However, his new concept, called the transfinite Hausdorff dimension is quite different in nature from ours, e.g. it takes ordinal numbers as values.

Next we say a few words about the main results and the organization of the chapter.

In Section 2.2 we investigate the basic properties of the topological Hausdorff dimension. Among others, we prove the following.

Theorem 2.4. $\dim_t(X) \leq \dim_{tH}(X) \leq \dim_H(X)$.

We also verify that $\dim_{tH} X$ satisfies some standard properties of a dimension, such as monotonicity, bi-Lipschitz invariance and countable stability for closed sets. Moreover, we check that this concept is genuinely new, since we show that $\dim_{tH} X$ cannot be expressed as a function of $\dim_t X$ and $\dim_H X$.

In Section 2.3 we compute $\dim_{tH} X$ for some classical fractals, like the Sierpiński triangle and carpet, the von Koch curve, etc. For example

Theorem 2.28. *Let T be the Sierpiński carpet. Then* $\dim_{tH}(T) = \frac{\log 6}{\log 3} = \frac{\log 2}{\log 3} + 1$.

(Note that $\dim_t T = 1$ and $\dim_H T = \frac{\log 8}{\log 3}$ while the Hausdorff dimension of the triadic Cantor set equals $\frac{\log 2}{\log 3}$.)

We also consider Kakeya sets (see [14] or [29]). Unfortunately, our methods do not give any useful information concerning the Kakeya Conjecture.

Theorem 2.30. *For every $d \in \mathbb{N}^+$ there exist a compact Kakeya set of topological Hausdorff dimension 1 in $\mathbb{R}^d$.*

Following [25] by T. W. Körner we prove somewhat more, since we essentially show that the generic element of a carefully chosen space is a Kakeya set of topological Hausdorff dimension 1.

We show that the trail of the Brownian motion almost surely (i.e. with probability 1) has topological Hausdorff dimension 1 in every dimension except perhaps 2 and 3. These two cases remain the most intriguing open problems of the chapter.

Problem 2.32. *Determine the almost sure topological Hausdorff dimension of the trail of the d-dimensional Brownian motion for $d = 2$ or 3.*

As our first application in Section 2.4 we generalize a result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. This limit set $M = M^{(p,n)}$ is a random Cantor set, which is constructed by dividing the unit square into $n \times n$ equal subsquares and keeping each of them independently with probability p, and then repeating the same procedure recursively for every subsquare. (See Section 2.4 for more details.)

Theorem 2.33 (Chayes-Chayes-Durrett, [10])**.** *There exists a critical probability $p_c = p_c^{(n)} \in (0, 1)$ such that if $p < p_c$ then M is totally disconnected almost surely, and if $p > p_c$ then M contains a nontrivial connected component with positive probability.*

It will be easy to see that this theorem is a special case of our next result.

Theorem 2.34. *For every $d \in [0, 2)$ there exists a critical probability $p_c^{(d)} = p_c^{(d,n)} \in (0, 1)$ such that if $p < p_c^{(d)}$ then $\dim_{tH} M \le d$ almost surely, and if $p > p_c^{(d)}$ then $\dim_{tH} M > d$ almost surely (provided $M \neq \emptyset$).*

Theorem 2.33 essentially says that certain curves show up at the critical probability, and our proof will show that even 'thick' families of curves show up, which roughly speaking means a 'Lipschitz copy' of $C \times [0, 1]$ with $\dim_H C > d - 1$.

We also give a numerical upper bound for $\dim_{tH} M$ which implies the following.

Corollary 2.50. *Almost surely*

$$\dim_{tH} M < \dim_H M \text{ or } M = \emptyset.$$

In Section 2.5 we answer Problem 1.3 as follows.

Corollary 2.72. *If K is a compact metric space with $\dim_t K > 0$ then we have $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ for the generic $f \in C(K)$.*

(If $\dim_t K = 0$ then the generic $f \in C(K)$ is one-to-one, thus every non-empty level set is of Hausdorff dimension 0.)

If K is also sufficiently homogeneous, e.g. self-similar then we can actually say more.

Corollary 2.74. *If K is a self-similar compact metric space with $\dim_t K > 0$ then $\dim_H f^{-1}(y) = \dim_{tH} K - 1$ for the generic $f \in C(K)$ and the generic $y \in f(K)$.*

In the course of the proofs, as a spin-off, we also provide a sequence of equivalent definitions of $\dim_{tH} K$ for compact metric spaces. Perhaps the most interesting one is the following.

Corollary 2.64. *If K is a compact metric space then $\dim_{tH} K$ is the smallest number d for which K can be covered by a finite family of compact sets of arbitrarily small diameter such that the set of points that are covered more than once has Hausdorff dimension at most $d - 1$.*

It can actually also be shown that in the equation $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ (for the generic $f \in C(K)$) the supremum is attained. On the other hand, one cannot say more in a sense, since there is a K such that for the generic $f \in C(K)$ there is a *unique* $y \in \mathbb{R}$ for which $\dim_H f^{-1}(y) = \dim_{tH} K - 1$. Moreover, in certain situations we can replace 'the generic $y \in f(K)$' with 'for every $y \in \operatorname{int} f(K)$' as in Kirchheim's theorem. The results of this last paragraph are to appear elsewhere, see [3].

Finally, in Section 2.6 we list some open problems.

1.3 Duality in LCA Polish groups

In this chapter we study a problem concerning duality between measure and category. Let G be a locally compact abelian (LCA) Polish group. Let $\mathcal{M}$ and $\mathcal{N}$ be the ideals of meager and null (with respect to Haar measure) subsets of G.

Definition 1.4. A bijection $F\colon G \to G$ is called an *Erdős-Sierpiński mapping* if

$$X \in \mathcal{N} \Leftrightarrow F[X] \in \mathcal{M} \quad \text{and} \quad X \in \mathcal{M} \Leftrightarrow F[X] \in \mathcal{N}.$$

Theorem 1.5 (Erdős–Sierpiński)**.** *Assume the Continuum Hypothesis. Then there exists an Erdős–Sierpiński mapping on* $\mathbb{R}$*.*

The existence of such a function is independent from ZFC. The main question of Chapter 3 is the following:

Is it consistent that there is an Erdős–Sierpiński mapping F that preserves addition, namely

$$\forall x, y \in G \quad F(x+y) = F(x) + F(y)?$$

This question is attributed to Ryll-Nardzewski in the case $G = \mathbb{R}$. Besides intrinsic interest, another motivation was the following: If this statement were consistent then the so called strong measure zero and strongly meager sets would consistently form isomorphic ideals. (For the definitions see [9].)

First T. Bartoszyński [7] gave a negative answer to the question in the case $G = 2^\omega$, then M. Kysiak proved this for $G = \mathbb{R}$ and answered the question of Ryll-Nardzewski, see [26], where he used and improved Bartoszyński's idea. We answer the general case, the goal of Chapter 3 is to prove the following theorem:

Theorem 3.1. *There is no addition preserving Erdős–Sierpiński mapping on any uncountable locally compact abelian Polish group.*

Let $(\varphi_{\mathcal{M}})$ denote the following statement (considered by Carlson in [9]): For every $S \in \mathcal{M}$ there exists a set $S' \in \mathcal{M}$ such that

$$\forall x_1, x_2 \in G \, \exists x \in G \quad (S + x_1) \cup (S + x_2) \subseteq S' + x.$$

Let $(\varphi_{\mathcal{N}})$ be the dual statement obtained by replacing $\mathcal{M}$ by $\mathcal{N}$.

If there exists an Erdős–Sierpiński mapping preserving addition then $(\varphi_{\mathcal{M}})$ and $(\varphi_{\mathcal{N}})$ are equivalent. In Section 3.2 we show that $(\varphi_{\mathcal{M}})$ holds in LCA Polish groups. In Section 3.3 we begin to show that $(\varphi_{\mathcal{N}})$ fails for all uncountable LCA Polish groups by reducing the general case to three special cases. Finally, in Section 3.4 we settle these three special cases.

1.4 The structure of rigid functions

An easy calculation shows that the exponential function $f(x) = e^x$ has the somewhat 'paradoxical' property that $cf(x)$ is a translate of $f(x)$ for every $c > 0$. It is also easy to see that every function of the form $a + be^{kx}$ has this property. This connection is also of interest from the point of view of functional equations. Cain, Clark and Rose

[8] introduced the notion of vertical rigidity, which we now formulate for functions of several variables.

Definition 4.1. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is called *vertically rigid*, if graph(cf) is isometric to graph(f) for all $c \in (0,\infty)$. (Clearly, $c \in \mathbb{R} \setminus \{0\}$ would be the same.)

The main results of Chapter 4 are the characterizations of the continuous vertically rigid functions of one and two variables. We need the following technical generalizations.

Definition 4.2. If C is a subset of $(0,\infty)$ and $\mathcal{G}$ is a set of isometries of $\mathbb{R}^{d+1}$ then we say that f is vertically rigid *for a set $C \subseteq (0,\infty)$ via elements of $\mathcal{G}$* if for every $c \in C$ there exists a $\varphi \in \mathcal{G}$ such that $\varphi(\text{graph}(f)) = \text{graph}(cf)$. (If we do not mention C or $\mathcal{G}$ then C is $(0,\infty)$ and $\mathcal{G}$ is the set of all isometries.)

In Section 4.1 we study the one-dimensional case. Clearly, every function of the form $a + bx$ is vertically rigid. D. Janković conjectured (see [8]) that the converse is also true for continuous functions.

Conjecture 4.3 (D. Janković)**.** *A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.*

In Subsection 4.1.2 we prove Janković's conjecture, even if we only assume that f is a continuous vertically rigid function for an uncountable set C. (Later C. Richter [33] proved a more general theorem.) We prove that if f is vertically rigid *via translations*, then also weaker regularity of f implies Jankovic's conjecture. In Subsection 4.1.3 we show that Janković's conjecture fails for Borel measurable functions. Our example also answers a question from [8] that asks whether every vertically rigid function is of the form $a + bx$ or $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g. In Subsection 4.1.4 we prove that every Lebesgue (Baire) measurable function that is vertically rigid *via translations* is of the form $a + be^{kx}$ *almost everywhere (on a co-meager set)*. The case of general isometries remains open. We also prove that in many situations the exceptional set can be removed. In Subsection 4.1.5 we define the notion of a rigid set, discuss how it is connected to the notion of a rigid function, and prove an ergodic theory type result.

In Section 4.2 we look at the two-dimensional case. The characterization of the continuous vertically rigid functions of two variables is the following.

Theorem 4.22. *A continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid if and only if after a suitable rotation around the z-axis $f(x,y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ $(a, b, d, k \in \mathbb{R},\ k \neq 0,\ s\colon \mathbb{R} \to \mathbb{R}$ continuous$)$.*

The structure of the proof will be as follows. First we check in Subsection 4.2.2 that functions of the above forms are rigid. (Of course, they are all continuous.) Then we start proving Theorem 4.22 in more and more general settings.

In Subsection 4.2.3 first we show that if all the isometries are horizontal translations then the vertically rigid function $f(x,y)$ is of the form $s(y)e^{kx}$ ($k \in \mathbb{R}$, $k \neq 0$, $s\colon \mathbb{R} \to \mathbb{R}$ continuous). The punchline here is that we can derive a simple functional equation from vertical rigidity (some sort of 'multiplicativity', see Lemma 4.9). Then we conclude this subsection by a completely algebraic proof showing that if we allow arbitrary translations then $f(x,y)$ is of the form $a + s(y)e^{kx}$ ($a, k \in \mathbb{R}$, $k \neq 0$, $s\colon \mathbb{R} \to \mathbb{R}$ continuous). Then we start working on the case of general isometries. The central idea is to consider the set S_f of directions of segments connecting pairs of points on graph(f) (see Definition 4.30). We collect the necessary properties of this set in Subsection 4.2.4. The set S_f has some sort of rigidity in that the transformation $f \mapsto cf$ distorts the shape of it, but the resulting set has to be isometric to the original one (see Definition 4.34 and Remark 4.35). Using these we determine the possible S_f's in Subsection 4.2.5, then in Subsection 4.2.6 we complete the proof by handling these cases using various methods.

In Section 4.3 we define the *horizontally rigid* functions, and characterize the functions that are horizontally rigid *via translations* in dimension one. We do not assume continuity, and analogous theorems hold in every dimension, see [4]. Later C. Richter [33] proved that the continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is horizontally rigid iff it is of the form $a + bx$. M. Elekes and the author of this thesis proved in [4] that the continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is horizontally rigid iff it is of the form $a + bx + dy$. The problem remains open in higher dimensions.

Finally, in Section 4.4 we collect the open questions of Chapter 4.

Chapter 2

The topological Hausdorff dimension

2.1 Preliminaries

Let (X, d) be a metric space. We denote by $\operatorname{cl} H$, $\operatorname{int} H$ and ∂H the closure, interior and boundary of a set H. For $x \in X$ and $H \subseteq X$ set $d(x, H) = \inf\{d(x, h) : h \in H\}$. Let $B(x, r)$ and $U(x, r)$ stand for the closed and open ball of radius r centered at x, respectively. More generally, for a set $H \subseteq X$ we define $B(H, r) = \{x \in X : d(x, H) \leq r\}$ and $U(H, r) = \{x \in X : d(x, H) < r\}$. The diameter of a set H is denoted by $\operatorname{diam} H$. We use the convention $\operatorname{diam} \emptyset = 0$. For two metric spaces (X, d_X) and (Y, d_Y) a function $f \colon X \to Y$ is *Lipschitz* if there exists a constant $C \in \mathbb{R}$ such that $d_Y(f(x_1), f(x_2)) \leq C \cdot d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. The smallest such constant C is called the Lipschitz constant of f and denoted by $\operatorname{Lip}(f)$. A function $f \colon X \to Y$ is called *bi-Lipschitz* if f is a bijection and both f and f^{-1} are Lipschitz. Let X be a metric space, $s \geq 0$ and $\delta > 0$, then

$$\begin{aligned}
\mathcal{H}^s_\infty(X) &= \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : X \subseteq \bigcup_{i=1}^{\infty} U_i\right\}, \\
\mathcal{H}^s_\delta(X) &= \inf\left\{\sum_{i=1}^{\infty} (\operatorname{diam} U_i)^s : X \subseteq \bigcup_{i=1}^{\infty} U_i, \forall i \operatorname{diam} U_i \leq \delta\right\}, \\
\mathcal{H}^s(X) &= \lim_{\delta \to 0+} \mathcal{H}^s_\delta(X).
\end{aligned}$$

The *Hausdorff dimension of* X is defined as

$$\dim_H X = \inf\{s \geq 0 : \mathcal{H}^s(X) = 0\}.$$

For more information on these concepts see [14] or [29].

Let X be a *complete* metric space. A set is *somewhere dense* if it is dense in a non-empty open set, and otherwise it is called *nowhere dense*. We say that $M \subseteq X$ is *meager* if it is a countable union of nowhere dense sets, and a set is called *co-meager* if its complement is meager. By Baire's Category Theorem co-meager sets are dense. It is not difficult to show that a set is co-meager iff it contains a dense G_δ set. We say that the generic element $x \in X$ has property $\mathcal{P}$, if $\{x \in X : x \text{ has property } \mathcal{P}\}$ is co-meager. The term 'typical' is also used instead of 'generic'. Our two main examples will be $X = C(K)$ endowed with the supremum metric (for some compact metric space K) and $X = \mathcal{K}$, that is, a certain subspace of the non-empty compact subsets of $\mathbb{R}^d$ endowed with the Hausdorff metric (i.e. $d_H(K_1, K_2) = \min\{r : K_1 \subseteq B(K_2, r) \text{ and } K_2 \subseteq B(K_1, r)\}$). See e.g. [23] for more on these concepts.

2.2 Basic properties of the topological Hausdorff dimension

Let X be a metric space. Since $\dim_t X = -1 \iff X = \emptyset \iff \dim_H X = -1$, we easily obtain

Fact 2.1. $\dim_{tH} X = 0 \iff \dim_t X = 0$.

As $\dim_H X$ is either -1 or at least 0, we obtain

Fact 2.2. *The topological Hausdorff dimension of a non-empty space is either* 0 *or at least* 1.

These two facts easily yield

Corollary 2.3. *Every metric space with a non-trivial connected component has topological Hausdorff dimension at least one.*

The next theorem states that the topological Hausdorff dimension is between the topological and the Hausdorff dimension.

Theorem 2.4. *For every metric space X*

$$\dim_t X \leq \dim_{tH} X \leq \dim_H X.$$

Proof. We can clearly assume that X is non-empty. It is well-known that $\dim_t X \leq \dim_H X$ (see e.g. [22]), which easily implies $\dim_t X \leq \dim_{tH} X$ using the definitions. The second inequality is obvious if $\dim_H X = \infty$. If $\dim_H X < 1$ then $\dim_t X = 0$ (since $\dim_t X \leq \dim_H X$ and $\dim_t X$ only takes integer values) and by Fact 2.1 we obtain $\dim_{tH} X = 0$, hence the second inequality holds. Therefore we may assume that $1 \leq \dim_H X < \infty$. The following lemma is basically [29, Thm. 7.7.]. It is only stated there in the special case $X = A \subseteq \mathbb{R}^n$, but the proof works verbatim for all metric spaces X.

Lemma 2.5. *Let X be a metric space and $f\colon X \to \mathbb{R}^m$ be Lipschitz. If $s > m$ then*

$$\int^\star \mathcal{H}^{s-m}\left(f^{-1}(y)\right)\, d\lambda_m(y) \leq c(m)\,\mathrm{Lip}(f)^m \mathcal{H}^s(X), \tag{2.1}$$

where $\int^\star$ denotes the upper Lebesgue integral, λ_m the m-dimensional Lebesgue measure and $c(m)$ is a finite constant depending only on m.

Now we return to the proof of Theorem 2.4. We fix $x_0 \in X$ and define $f\colon X \to \mathbb{R}$ by $f(x) = d_X(x, x_0)$. Using the triangle inequality it is easy to see that f is Lipschitz with $\mathrm{Lip}(f) \leq 1$. We fix $n \in \mathbb{N}^+$ and apply Lemma 2.5 for f and $s = \dim_H X + \frac{1}{n} > 1 = m$. Hence

$$\int^\star \mathcal{H}^{s-1}(f^{-1}(y))\, d\lambda_1(y) \leq c(1)\mathcal{H}^s(X) = 0.$$

Thus $\mathcal{H}^{s-1}(f^{-1}(y)) = \mathcal{H}^{\dim_H X + \frac{1}{n} - 1}(f^{-1}(y)) = 0$ holds for a.e. $y \in \mathbb{R}$. Since this is true for all $n \in \mathbb{N}^+$, we obtain that $\dim_H f^{-1}(y) \leq \dim_H(X) - 1$ for a.e. $y \in \mathbb{R}$. From the definition of f it follows that $\partial U(x_0, y) \subseteq f^{-1}(y)$. Hence there is a neighborhood basis of x_0 with boundaries of Hausdorff dimension at most $\dim_H(X) - 1$, and this is true for all $x_0 \in X$, so there is a basis with boundaries of Hausdorff dimension at most $\dim_H(X) - 1$. By the definition of the topological Hausdorff dimension this implies $\dim_{tH} X \leq \dim_H X$. □

There are some elementary properties one expects from a notion of dimension. Now we verify some of these for the topological Hausdorff dimension.

Extension of the classical dimension. Theorem 2.4 implies that the topological Hausdorff dimension of a countable set equals zero, moreover, for open subspaces of $\mathbb{R}^d$ and for smooth d-dimensional manifolds the topological Hausdorff dimension equals d.

Monotonicity. Let $X \subseteq Y$. If $\mathcal{U}$ is a basis in Y then $\mathcal{U}_X = \{U \cap X : U \in \mathcal{U}\}$ is a basis in X, and $\partial_X(U \cap X) \subseteq \partial_Y U$ holds for all $U \in \mathcal{U}$. This yields

Fact 2.6 (Monotonicity)**.** *If $X \subseteq Y$ are metric spaces then* $\dim_{tH} X \leq \dim_{tH} Y$.

Bi-Lipschitz invariance. First we prove that the topological Hausdorff dimension does not increase under Lipschitz homeomorphisms. An easy consequence of this that our dimension is bi-Lipschitz invariant, and does not increase under an injective Lipschitz map on a compact space. After obtaining corollaries of Theorem 2.7 we give some examples illustrating the necessity of certain conditions in this theorem and its corollaries.

Theorem 2.7. *Let X, Y be metric spaces. If $f \colon X \to Y$ is a Lipschitz homeomorphism then* $\dim_{tH} Y \leq \dim_{tH} X$.

Proof. Since f is a homeomorphism, if $\mathcal{U}$ is a basis in X then $\mathcal{V} = \{f(U) : U \in \mathcal{U}\}$ is a basis in Y, and $\partial f(U) = f(\partial U)$ for all $U \in \mathcal{U}$. The Lipschitz property of f implies that $\dim_H \partial V = \dim_H \partial f(U) = \dim_H f(\partial U) \leq \dim_H \partial U$ for all $V = f(U) \in \mathcal{V}$. Thus $\dim_{tH} Y \leq \dim_{tH} X$. □

This immediately implies the following two statements.

Corollary 2.8 (Bi-Lipschitz invariance)**.** *Let X, Y be metric spaces. If $f \colon X \to Y$ is bi-Lipschitz then* $\dim_{tH} X = \dim_{tH} Y$.

Corollary 2.9. *If K is a compact metric space, and $f \colon K \to Y$ is* one-to-one *Lipschitz then* $\dim_{tH} f(K) \leq \dim_{tH} K$.

The following example shows that we cannot drop injectivity here. First we need a well-known lemma.

Lemma 2.10. *Let $M \subseteq \mathbb{R}$ be measurable with positive Lebesgue measure. Then there exists a Lipschitz onto map $f \colon M \to [0,1]$.*

Proof. Let us choose a compact set $C \subseteq M$ of positive Lebesgue measure. Define $f \colon M \to [0,1]$ by

$$f(x) = \frac{\lambda\left((-\infty, x) \cap C\right)}{\lambda(C)},$$

where λ denotes the one-dimensional Lebesgue measure. Then it is not difficult to see that f is Lipschitz (with $\mathrm{Lip}(f) \leq \frac{1}{\lambda(C)}$) and $f(C) = [0,1]$. □

Example 2.11. Let $K \subseteq \mathbb{R}$ be a Cantor set (that is, a set homeomorphic to the middle-thirds Cantor set) of positive Lebesgue measure. By Fact 2.1, $\dim_{tH} K = \dim_t K = 0$. Using Lemma 2.10 there is a Lipschitz map $f \colon K \to [0,1]$ such that $f(K) = [0,1]$. By Theorem 2.4, $\dim_{tH}[0,1] = 1$, hence $\dim_{tH} K = 0 < 1 = \dim_{tH}[0,1] = \dim_{tH} f(K)$.

The next example shows that Corollary 2.9 does not hold without the assumption of compactness. We even have a separable metric counterexample.

Example 2.12. Let C be the middle-thirds Cantor set, and $f\colon C\times C \to [0,2]$ be defined by $f(x,y) = x+y$. It is well-known and easy to see that f is Lipschitz and $f(C\times C) = [0,2]$. Therefore one can select a subset $X \subseteq C\times C$ such that $f|_X$ is a bijection from X onto $[0,2]$. Then X is separable metric. Monotonicity and $\dim_t(C\times C) = 0$ imply $\dim_{tH} X \leq \dim_{tH}(C\times C) = 0$. Therefore, f is one-to-one and Lipschitz on X but $\dim_{tH} X = 0 < 1 = \dim_{tH}[0,2] = \dim_{tH} f(X)$.

Our last example shows that the topological Hausdorff dimension is not invariant under homeomorphisms. Not even for compact metric spaces.

Example 2.13. Let $C_1, C_2 \subseteq \mathbb{R}$ be Cantor sets such that $\dim_H C_1 \neq \dim_H C_2$. We will see in Theorem 2.19 that $\dim_{tH}(C_i \times [0,1]) = \dim_H C_i + 1$ for $i = 1,2$. Hence $C_1 \times [0,1]$ and $C_2 \times [0,1]$ are homeomorphic compact metric spaces whose topological Hausdorff dimensions disagree.

Stability and countable stability. As the following example shows, similarly to the case of topological dimension, stability does not hold for non-closed sets. That is, $X = \bigcup_{n=1}^{k} X_n$ does not imply $\dim_{tH} X = \max_{1\leq n\leq k} \dim_{tH} X_n$.

Example 2.14. Theorem 2.4 implies $\dim_{tH}(\mathbb{R}) = 1$, and Fact 2.1 yields $\dim_{tH}(\mathbb{Q}) = \dim_t(\mathbb{Q}) = 0$ and $\dim_{tH}(\mathbb{R}\setminus\mathbb{Q}) = \dim_t(\mathbb{R}\setminus\mathbb{Q}) = 0$. Thus $\dim_{tH}\mathbb{R} = 1 > 0 = \max\{\dim_{tH}(\mathbb{Q}), \dim_{tH}(\mathbb{R}\setminus\mathbb{Q})\}$, and therefore stability fails.

As a corollary, we now show that as opposed to the case of Hausdorff (and packing) dimension, there is no reasonable family of measures inducing the topological Hausdorff dimension. Let us say that a 1-parameter family of measures $\{\mu^s\}_{s\geq 0}$ is *monotone* if $\mu^s(A) = 0$, $s < t$ implies $\mu^t(A) = 0$. The family of Hausdorff (or packing) measures certainly satisfies this criterion. It is not difficult to see that monotonicity implies that the induced notion of dimension, that is, $\dim A = \inf\{s : \mu^s(A) = 0\}$ is countably stable. Hence we obtain

Corollary 2.15. *There is no monotone 1-parameter family of measures* $\{\mu^s\}_{s\geq 0}$ *such that* $\dim_{tH} A = \inf\{s : \mu^s(A) = 0\}$.

However, just like in the case of topological dimension, even countable stability holds for *closed* sets.

Theorem 2.16 (Countable stability for closed sets)**.** *Let X be a separable metric space and $X = \bigcup_{n\in\mathbb{N}} X_n$, where X_n $(n \in \mathbb{N})$ are closed subsets of X. Then* $\dim_{tH} X = \sup_{n\in\mathbb{N}} \dim_{tH} X_n$.

Proof. Monotonicity clearly implies $\dim_{tH} X \geq \sup_{n\in\mathbb{N}} \dim_{tH} X_n$. For the other direction we may assume $\sup_{n\in\mathbb{N}} \dim_{tH} X_n < \infty$. Let $d > \sup_{n\in\mathbb{N}} \dim_{tH} X_n$ be arbitrary. Assume $\mathcal{U}_n$, $n \in \mathbb{N}$ is a countable basis of X_n such that $\dim_H \partial_{X_n} U \leq d-1$ for all $n \in \mathbb{N}$ and $U \in \mathcal{U}_n$. Let $Y = \bigcup\{\partial_{X_n} U : n \in \mathbb{N}, U \in \mathcal{U}_n\}$. By countable stability of the Hausdorff dimension, $\dim_H Y \leq d-1$. Using the definition of the topological dimension we obtain $\dim_t(X_n \setminus Y) = 0$ for all $n \in \mathbb{N}$. The set $X_n \setminus Y$ is closed in the separable metric space $X \setminus Y$, and $X \setminus Y = \bigcup_{n\in\mathbb{N}}(X_n \setminus Y)$. By the sum theorem for topological dimension 0, see [12, 1.3.1], $\dim_t(X \setminus Y) = 0$.

Let us fix an open set $V \subseteq X$ and a point $x \in V$. Using that $X \setminus Y$ is a separable subspace of X with topological dimension 0, by the separation theorem for topological dimension zero [12, 1.2.11.] there is a so-called partition between x and $X \setminus V$ disjoint from $X \setminus Y$. This means that there exist disjoint open sets $U, U' \subseteq X$ such that $x \in U$, $X \setminus V \subseteq U'$ and $(X \setminus (U \cup U')) \cap (X \setminus Y) = \emptyset$. In particular, $x \in U \subseteq V$. Moreover, $\partial_X U \cap (X \setminus Y) = \emptyset$, so $\partial_X U \subseteq Y$, thus $\dim_H \partial_X U \leq \dim_H Y \leq d-1$. By the definition of topological Hausdorff dimension we obtain $\dim_{tH} X \leq d$. As $d > \sup_{n\in\mathbb{N}} \dim_{tH} X_n$ was arbitrary, the proof is complete. □

Corollary 2.17. *The same holds for F_σ sets, as well.*

Products. Now we investigate products from the point of view of topological Hausdorff dimension. By product of two metric spaces we will always mean the l^2-product, that is,

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = \sqrt{d_X^2(x_1, x_2) + d_Y^2(y_1, y_2)}.$$

First we recall a well-known statement, see [14, Chapters 3 and 7] for the definitions and the proof.

Lemma 2.18. *Let X, Y be non-empty metric spaces such that $\dim_H Y = \overline{\dim}_B Y$, where $\overline{\dim}_B$ is the upper box-counting dimension. Then*

$$\dim_H(X \times Y) = \dim_H X + \dim_H Y.$$

Now we prove our next theorem which provides a large class of sets for which the topological Hausdorff dimension and the Hausdorff dimension coincide.

Theorem 2.19. *Let X be a non-empty separable metric space. Then*

$$\dim_{tH}(X \times [0,1]) = \dim_H (X \times [0,1]) = \dim_H X + 1.$$

Proof. Applying Lemma 2.18 for $Y = [0,1]$ we obtain $\dim_H (X \times [0,1]) = \dim_H X + \dim_H[0,1] = \dim_H X + 1$. In addition, Theorem 2.4 follows that $\dim_{tH}(X \times [0,1]) \leq \dim_H (X \times [0,1])$. For the opposite inequality we need the following lemma.

Lemma 2.20. *Let X be a non-empty separable metric space and let $d < \dim_H X$ be fixed. Then there exists $x_d \in X$ such that $\dim_H U(x_d, r) \geq d$ holds for every $r > 0$.*

Proof of the Lemma. Assume, on the contrary, that for all $x \in X$ there is an $r_x > 0$ such that $\dim_H U(x, r_x) < d$. Since X is separable, by the Lindelöf property we can select a countable subcover $\{U(x_n, r_n)\}_{n\in\mathbb{N}}$ of the cover $\{U(x, r_x)\}_{x\in X}$. By countable stability of the Hausdorff dimension $\dim_H X = \sup_{n\in\mathbb{N}} \dim_H U(x_n, r_n) \leq d$, which is a contradiction. □

We now return to the proof of Theorem 2.19. For a fixed $d < \dim_H X$ assume that $x_d \in X$ is given as in the lemma. Let $\mathcal{U}$ be a basis in $X \times [0,1]$ and $\mathrm{pr}_X \colon X \times [0,1] \to X$, $\mathrm{pr}_X(x,y) = x$. There exists $U_d \in \mathcal{U}$ such that $(x_d, 1) \in U_d$ and $U_d \cap \left(X \times \left[0, \frac{1}{2}\right]\right) = \emptyset$. Then there is an $r_d > 0$ such that $U(x_d, r_d) \times [1 - r_d, 1] \subseteq U_d$. For every $x \in U(x_d, r_d)$ we have $(x, 0) \notin U_d$ and $(x, 1) \in U_d$, hence $\partial U_d \cap (x \times [0,1]) \neq \emptyset$. Thus $U(x_d, r_d) \subseteq \mathrm{pr}_X(\partial U_d)$. Projections do not increase the Hausdorff dimension, therefore $\dim_H \partial U_d \geq \dim_H U(x_d, r_d) \geq d$. This is valid for all $d < \dim_H X$, so $\sup_{U\in\mathcal{U}} \dim_H \partial U \geq \dim_H X$ for all basis $\mathcal{U}$, thus $\dim_{tH}(X \times [0,1]) \geq \dim_H X + 1$ by the definition of topological Hausdorff dimension. □

Remark 2.21. We cannot drop separability here. Indeed, if X is an uncountable discrete metric space then it is not difficult to see that $\dim_{tH}(X \times [0,1]) = 1$ and $\dim_H(X \times [0,1]) = \dim_H X = \infty$.

Separability is a rather natural assumption throughout the chapter. First, the Hausdorff dimension is only meaningful in this context (it is always infinite for non-separable spaces), secondly for the theory of topological dimension this is the most usual framework.

Corollary 2.22. *If X is a non-empty separable metric space then*

$$\dim_{tH}\left(X \times [0,1]^d\right) = \dim_H \left(X \times [0,1]^d\right) = \dim_H X + d.$$

The possible values of $(\dim_t X, \dim_{tH} X, \dim_H X)$**.**

The following theorem provides a complete description of the possible values of the triple $(\dim_t X, \dim_{tH} X, \dim_H X)$. Moreover, all possible values can be realized by compact spaces as well.

Theorem 2.23. *For a triple $(d, s, t) \in [0, \infty]^3$ the following are equivalent.*

(i) There exists a compact metric space K such that $\dim_t K = d$, $\dim_{tH} K = s$, and $\dim_H K = t$.

(ii) There exists a separable metric space X such that $\dim_t X = d$, $\dim_{tH} X = s$, and $\dim_H X = t$.

(iii) There exists a metric space X such that $\dim_t X = d$, $\dim_{tH} X = s$, and $\dim_H X = t$.

(iv) $d = s = t = -1$, or $d = s = 0$, $t \in [0, \infty]$, or $d \in \mathbb{N}^+ \cup \{\infty\}$, $s, t \in [1, \infty]$, $d \leq s \leq t$.

Proof. The implications $(i) \implies (ii)$ and $(ii) \implies (iii)$ are obvious, and $(iii) \implies (iv)$ can easily be checked using Fact 2.1 and Theorem 2.4.

It remains to prove that $(iv) \implies (i)$. First, the empty set takes care of the case $d = s = t = -1$. Let now $d = s = 0$, $t \in [0, \infty]$. For $t \in [0, \infty]$ let K_t be a Cantor set with $\dim_H K_t = t$. Such sets are well-known to exist already in $[0, 1]^n$ for large enough n in case $t < \infty$, whereas if C is the middle-thirds Cantor set then $C^{\mathbb{N}}$ is such a set for $t = \infty$. Then clearly $\dim_t K_t = \dim_{tH} K_t = 0$ and $\dim_H K_t = t$, so we are done with this case.

Finally, let $d \in \mathbb{N}^+ \cup \{\infty\}$, $s, t \in [1, \infty]$, $d \leq s \leq t$. We may assume $d < \infty$, otherwise the Hilbert cube provides a suitable example. (Indeed, clearly $\dim_t [0, 1]^{\mathbb{N}} = \dim_{tH} [0, 1]^{\mathbb{N}} = \dim_H [0, 1]^{\mathbb{N}} = \infty$.) Define $K_{d,s,t} = (K_{s-d} \times [0, 1]^d) \cup K_t$ (this can be understood as the disjoint sum of metric spaces, but we may also assume that all these spaces are in the Hilbert cube, so the union is well defined). Since $\dim_t (X \times Y) \leq \dim_t X + \dim_t Y$ for non-empty spaces (see e.g. [12]), we obtain $\dim_t (K_{s-d} \times [0, 1]^d) = 0 + d = d$. Hence, by the stability of the topological dimension for closed sets, $\dim_t K_{d,s,t} = \max \left\{ \dim_t \left(K_{s-d} \times [0, 1]^d \right), \dim_t K_t \right\} = \max\{d, 0\} = d$. Using Corollary 2.22 and the stability of the topological Hausdorff dimension for closed sets we infer that $\dim_{tH} K_{d,s,t} = \max \left\{ \dim_{tH} \left(K_{s-d} \times [0, 1]^d \right), \dim_{tH} K_t \right\} = \max\{s - d + d, 0\} = s$. Again by Corollary 2.22 and by the stability of the Hausdorff dimension we obtain that

$\dim_H K_{d,s,t} = \max\left\{\dim_H\left(K_{s-d} \times [0,1]^d\right), \dim_H K_t\right\} = \max\{s-d+d, t\} = \max\{s,t\} = t$. This completes the proof. $\square$

The topological Hausdorff dimension is not a function of the topological and the Hausdorff dimension.

As a particular case of the above theorem we obtain that there are compact metric spaces X and Y such that $\dim_t X = \dim_t Y$ and $\dim_H X = \dim_H Y$ but $\dim_{tH} X \neq \dim_{tH} Y$. This immediately implies the following, which shows that the topological Hausdorff dimension is indeed a genuinely new concept.

Corollary 2.24. *$\dim_{tH} X$ cannot be calculated form $\dim_t X$ and $\dim_H X$, even for compact metric spaces.*

2.3 Calculating the topological Hausdorff dimension

2.3.1 Some classical fractals

First we present certain natural examples of compact sets K with $\dim_t K = \dim_{tH} K < \dim_H K$. Let S be the Sierpiński triangle, then it is well-known that $\dim_t S = 1$ and $\dim_H S = \frac{\log 3}{\log 2}$.

Theorem 2.25. *Let S be the Sierpiński triangle. Then $\dim_{tH}(S) = 1$.*

Proof. Let $\varphi_i \colon \mathbb{R}^2 \to \mathbb{R}^2$ $(i = 1,2,3)$ be the three similitudes with ratio $1/2$ for which $S = \bigcup_{i=1}^3 \varphi_i(S)$. Sets of the form $\varphi_{i_n} \circ \cdots \circ \varphi_{i_1}(S)$, $n \in \mathbb{N}$, $j \in \{1,\ldots,n\}$, $i_j \in \{1,2,3\}$ are called the elementary pieces of S. It is not difficult to see that

$$\mathcal{U} = \{\operatorname{int}_S H : H \text{ is a finite union of elementary pieces of } S\}$$

is a basis of S such that $\#\partial_S U$ is finite for every $U \in \mathcal{U}$. Therefore $\dim_H \partial_S U \leq 0$, and hence $\dim_{tH} S \leq 1$. On the other hand, S contains a line segment, therefore $\dim_{tH} S \geq \dim_{tH}[0,1] = 1$ by monotonicity. $\square$

Now we turn to the von Koch snowflake curve K. Recall that $\dim_t K = 1$ and $\dim_H K = \frac{\log 4}{\log 3}$.

Fact 2.26. *If K is homeomorphic to $[0,1]$ then $\dim_{tH} K = 1$.*

Proof. By Corollary 2.3 we obtain that $\dim_{tH} K \geq 1$. On the other hand, since K is homeomorphic to $[0,1]$, there is a basis in K such that $\#\partial U \leq 2$ for every $U \in \mathcal{U}$. Thus $\dim_{tH} K \leq 1$. $\square$

Corollary 2.27. *Let K be the von Koch curve. Then* $\dim_{tH} K = 1$.

Next we take up a natural example of a compact set K with $\dim_t K < \dim_{tH} K < \dim_H K$. Let T be the Sierpiński carpet, then it is well-known that $\dim_t T = 1$ and $\dim_H T = \frac{\log 8}{\log 3}$.

Theorem 2.28. *Let T be the Sierpiński carpet. Then* $\dim_{tH}(T) = \frac{\log 2}{\log 3} + 1 = \frac{\log 6}{\log 3}$.

Proof. Let C denote the middle-thirds Cantor set. Observe that $C \times [0,1] \subseteq T$. Then monotonicity and Theorem 2.19 yield $\dim_{tH} T \geq \dim_{tH}(C \times [0,1]) = \dim_H C + 1 = \frac{\log 2}{\log 3} + 1$.

Let us now prove the opposite inequality. For $n \in \mathbb{N}$ and $i = 1, \dots, 3^n$ let $z_i^n = \frac{2i-1}{2(3^n)}$. Then clearly

$$\{z_i^n : n \in \mathbb{N},\, i \in \{1, \dots, 3^n\}\}$$

is dense in $[0,1]$. Let L be a horizontal line defined by an equation of the form $y = z_i^n$ or a vertical line defined by $x = z_i^n$. It is easy to see that $L \cap T$ consists of finitely many sets geometrically similar to the middle-thirds Cantor set. Using these lines it is not difficult to construct a rectangular basis $\mathcal{U}$ of T such that $\dim_H \partial_T U = \frac{\log 2}{\log 3}$ for every $U \in \mathcal{U}$, and hence $\dim_{tH} T \leq \frac{\log 2}{\log 3} + 1$. $\square$

Finally we remark that, by Theorem 2.19, $K = C \times [0,1]$ (where C is the middle-thirds Cantor set) is a natural example of a compact set with $\dim_t K < \dim_{tH} K = \dim_H K$.

2.3.2 Kakeya sets

Definition 2.29. A subset of $\mathbb{R}^d$ is called a *Kakeya set* if it contains a non-degenerate line segment in every direction (some authors call these sets *Besicovitch sets*).

According to a surprising classical result, Kakeya sets of Lebesgue measure zero exist. However, one of the most famous conjectures in analysis is the Kakeya Conjecture stating that every Kakeya set in $\mathbb{R}^d$ has Hausdorff dimension d. This is known to hold only in dimension at most 2 so far, and a solution already in $\mathbb{R}^3$ would have a huge impact on numerous areas of mathematics.

It would be tempting to attack the Kakeya Conjecture using $\dim_{tH} K \leq \dim_H K$, but the following theorem, the main theorem of this section will show that unfortunately we cannot get anything non-trivial this way.

Theorem 2.30. *There exists a Kakeya set $K \subseteq \mathbb{R}^d$ of topological Hausdorff dimension 1 for every integer $d \geq 1$.*

This result is of course sharp, since if a set contains a line segment then its topological Hausdorff dimension is at least 1.

We will actually prove somewhat more, since we will essentially show that the generic element of a carefully chosen space is a Kakeya set of topological Hausdorff dimension 1. This idea, as well as most of the others in this subsection are already present in [25] by T. W. Körner. However, he only works in the plane and his space slightly differs from ours. For the sake of completeness we provide the rather short proof details.

Let $(\mathcal{K}, d_H)$ be the set of compact subsets of $\mathbb{R}^{d-1} \times [0,1]$ endowed with the Hausdorff metric, that is for each $K_1, K_2 \in \mathcal{K}$

$$d_H(K_1, K_2) = \min \{r : K_1 \subseteq B(K_2, r) \text{ and } K_2 \subseteq B(K_1, r)\},$$

where $B(K, r) = \left\{x \in \mathbb{R}^{d-1} \times [0,1] : \operatorname{dist}(x, K) \leq r\right\}$. It is well-known that $(\mathcal{K}, d_H)$ is a complete metric space, see e.g. [23].

Let

$$\Gamma = \{(x_1, \ldots, x_{d-1}, 1) : 1/2 \leq x_i \leq 1, \quad i = 1, \ldots, d-1\}$$

denote a subset of directions in $\mathbb{R}^d$. A closed line segment w connecting $\mathbb{R}^{d-1} \times \{0\}$ and $\mathbb{R}^{d-1} \times \{1\}$ is called a standard segment.

Let us denote by $\mathcal{F} \subseteq \mathcal{K}$ the system of those compact sets in $\mathbb{R}^{d-1} \times [0,1]$ in which for each $v \in \Gamma$ we can find a standard segment w parallel to v. First we show that $\mathcal{F}$ is closed in $\mathcal{K}$. Let us assume that $F_n \in \mathcal{F}$, $K \in \mathcal{K}$ and $F_n \to K$ with respect to d_H. We have to show that $K \in \mathcal{F}$. Let $v \in \Gamma$ be arbitrary. Since $F_n \in \mathcal{F}$, there exists a $w_n \subseteq F_n$ parallel to v for every n. It is easy to see that $\bigcup_{n \in \mathbb{N}} F_n$ is bounded, hence we can choose a subsequence n_k such that w_{n_k} is convergent with respect to d_H. But then clearly $w_{n_k} \to w$ for some standard segment $w \subseteq K$, and w is parallel to v. Hence $K \in \mathcal{F}$ indeed.

Therefore, $(\mathcal{F}, d_H)$ is a complete metric space and hence we can use Baire category arguments.

The next lemma is based on [25, Thm. 3.6.].

Lemma 2.31. *The generic set in $(\mathcal{F}, d_H)$ is of topological Hausdorff dimension 1.*

Proof. The rational cubes form a basis of $\mathbb{R}^d$, and their boundaries are covered by the rational hyperplanes orthogonal to one of the usual basis vectors of $\mathbb{R}^d$. Therefore, it suffices to show that if S is a fixed hyperplane orthogonal to one of the usual basis vectors then $\{F \in \mathcal{F} : \dim_H(F \cap S) = 0\}$ is co-meager.

For $n \in \mathbb{N}^+$ define

$$\mathcal{F}_n = \left\{ F \in \mathcal{F} : \mathcal{H}^{\frac{1}{n}}_{\frac{1}{n}}(F \cap S) < \frac{1}{n} \right\}.$$

In order to show that $\{F \in \mathcal{F} : \dim_H(F \cap S) = 0\} = \bigcap_{n \in \mathbb{N}^+} \mathcal{F}_n$ is co-meager, it is enough to prove that each $\mathcal{F}_n$ contains a dense open set.

For $p \in \mathbb{R}^d$, $v \in \Gamma$ and $0 < \alpha < \pi/2$ we denote by $C(p, v, \alpha)$ the following doubly infinite closed cone

$$C(p, v, \alpha) = \{x \in \mathbb{R}^d : \text{ the angle between the lines of } v \text{ and } x - p \text{ is at most } \alpha\}.$$

We denote by $V(C(p, v, \alpha))$ the set of those vectors $u = (u_1, ..., u_{d-1}, 1)$ for which there is a line in $\operatorname{int}(C(p, v, \alpha)) \cup \{p\}$ parallel to u. Then $V(C(p, v, \alpha))$ is relatively open in $\mathbb{R}^{d-1} \times \{1\}$.

The sets of the form $C'(p, v, \alpha) = C(p, v, \alpha) \cap \left(\mathbb{R}^{d-1} \times [0, 1]\right)$ will be called truncated cones, and the system of truncated cones will be denoted by $\mathcal{C}'$. A truncated cone $C'(p, v, \alpha)$ is S-compatible if either $C'(p, v, \alpha) \cap S = \{p\}$, or $C'(p, v, \alpha) \cap S = \emptyset$. The set of S-compatible truncated cones is denoted by $\mathcal{C}'_S$. Define $\mathcal{F}_S$ as the set of those $F \in \mathcal{F}$ that can be written as the union of finitely many S-compatible truncated cones and finitely many points in $\mathbb{R}^{d-1} \times [0, 1]$.

Next we check that $\mathcal{F}_S$ is dense in $\mathcal{F}$.

Suppose $F \in \mathcal{F}$ is arbitrary and $\varepsilon > 0$ is given. First choose finitely many points $\{y_i\}_{i=1}^t$ in F such that $F \subseteq B\left(\{y_i\}_{i=1}^t, \varepsilon\right)$. Let $v \in \Gamma$ be arbitrary, then there exists a standard segment $w_v \subseteq F$ parallel to v. By the choice of S and Γ, clearly $w_v \nsubseteq S$, hence we can choose p_v and α_v such that $C'(p_v, v, \alpha_v) \in \mathcal{C}'_S$ and $d_H(C'(p_v, v, \alpha_v), w_v) \leq \varepsilon$. Obviously $v \in V(C(p_v, v, \alpha_v))$, so $\{V(C(p_v, v, \alpha_v))\}_{v \in \Gamma}$ is an open cover of the compact set Γ. Therefore, there are $\{C'(p_{v_i}, v_i, \alpha_{v_i})\}_{i=1}^m$ in $\mathcal{C}'_S$ such that $\Gamma \subseteq \bigcup_{i=1}^m V(C(p_{v_i}, v_i, \alpha_{v_i}))$. Put $F' = \bigcup_{i=1}^m C'(p_{v_i}, v_i, \alpha_{v_i}) \cup \{y_1, \ldots, y_t\}$, then $F' \in \mathcal{F}_S$. It is easy to see that $\bigcup_{i=1}^m C'(p_{v_i}, v_i, \alpha_{v_i}) \subseteq B(F, \varepsilon)$, and combining this with $\{y_i\}_{i=1}^t \subseteq F$ we obtain that $F' \subseteq B(F, \varepsilon)$. By the choice of $\{y_i\}_{i=1}^t$ we also have $F \subseteq B(F', \varepsilon)$. Thus $d_H(F, F') \leq \varepsilon$.

Now using our dense set $\mathcal{F}_S$ we verify that $\mathcal{F}_n$ contains a dense open set $\mathcal{U}$. We construct for all $F_0 \in \mathcal{F}_S$ a ball in $\mathcal{F}_n$ centered at F_0. By the definition of S-compatibility $F_0 \cap S$ is finite. Hence we can easily choose a relatively open set $U_0 \subseteq S$ such that $F_0 \cap S \subseteq U_0$ and $\mathcal{H}^{\frac{1}{n}}_{\frac{1}{n}}(U_0) < \frac{1}{n}$. Let us define

$$\mathcal{U} = \{F \in \mathcal{F} : F \cap S \subseteq U_0\}.$$

Clearly $F_0 \in \mathcal{U}$, $\mathcal{U} \subseteq \mathcal{F}_n$ and it is easy to see that $\mathcal{U}$ is open in $\mathcal{F}$. This completes the proof. $\square$

From this we obtain the main theorem of the section as follows.

Proof of Theorem 2.30. The above lemma implies that we can choose $F_0 \in \mathcal{F}$ such that $\dim_{tH} F_0 = 1$. Then F_0 contains a line segment in every direction of Γ, hence we can choose finitely many isometric copies of it, $\{F_i\}_{i=1}^n$ such that the compact set $K = \bigcup_{i=0}^n F_i$ contains a line segment in every direction. By the Lipschitz invariance of the topological Hausdorff dimension $\dim_{tH} F_i = \dim_{tH} F_0$ for all i, and by the stability of the topological Hausdorff dimension for closed sets $\dim_{tH} K = 1$. $\square$

2.3.3 Brownian motion

One of the most important stochastic processes is the Brownian motion (see e.g. [30]). Its trail and graph also serve as important examples of fractal sets in geometric measure theory. Since the graph is always homeomorphic to $[0, \infty)$, Fact 2.26 and countable stability for closed sets yield that its topological Hausdorff dimension is 1. Hence we focus on the trail only.

Each statement in this paragraph is to be understood to hold with probability 1 (almost surely). Clearly, in dimension 1 the trail is a non-degenerate interval, so it has topological Hausdorff dimension 1. Moreover, if the dimension is at least 4 then the trail has no multiple points ([30]), so it is homeomorphic to $[0, \infty)$, which in turn implies as above that the trail has topological Hausdorff dimension 1 again. However, the following question is open.

Problem 2.32. *Let $d = 2$ or 3. Determine the almost sure topological Hausdorff dimension of the trail of the d-dimensional Brownian motion.*

2.4 Application I: Mandelbrot's fractal percolation process

In this section we take up one of the most important random fractals, the limit set M of the fractal percolation process defined by Mandelbrot in [28].

His original motivation was that this model captures certain features of turbulence, but then this random set turned out to be very interesting in its own right. For example, M serves as a very powerful tool for calculating Hausdorff dimension. Indeed, J. Hawkes has shown ([19]) that for a fixed Borel set B in the unit square (or analogously in higher dimensions) $M \cap B = \emptyset$ almost surely iff the sum of the co-dimensions exceeds 2, (that is, $(2 - \dim_H M) + (2 - \dim_H B) > 2$), and this formula can be used in certain applications to determine $\dim_H B$. Moreover, it can be shown that the trail of the Brownian motion is so called *intersection-equivalent* to a percolation fractal (roughly speaking, they intersect the same sets with positive probability), and this can be used to deduce numerous dimension related results about the Brownian motion, see the works of Y. Peres, e.g. in [30].

Let us now formally describe the fractal percolation process. Let $p \in (0,1)$ and $n \geq 2$, $n \in \mathbb{N}$ be fixed. Set $M_0 = M_0^{(p,n)} = [0,1]^2$. We divide the unit square into n^2 equal closed subsquares of side-length $1/n$ in the natural way. We keep each subsquare independently with probability p (and erase it with probability $1-p$), and denote by $M_1 = M_1^{(p,n)}$ the union of the kept subsquares. Then each square in M_1 is divided into n^2 squares of side-length $1/n^2$, and we keep each of them independently (and also independently of the earlier choices) with probability p, etc. After k steps let $M_k = M_k^{(p,n)}$ be the union of the kept k^{th} level squares with side-length $1/n^k$. Let

$$M = M^{(p,n)} = \bigcap_{k=1}^{\infty} M_k. \tag{2.2}$$

The process we have just described is called *Mandelbrot's fractal percolation process*, and M is called its *limit set*.

Percolation fractals are not only interesting from the point of view of turbulence and fractal geometry, but they are also closely related to the (usual, graph-theoretic) percolation theory. In case of the fractal percolation the role of the clusters is played by the connected components. Our starting point will be the following celebrated theorem. Recall that a space is *totally disconnected* if every connected component is a singleton.

Theorem 2.33 (Chayes-Chayes-Durrett, [10])**.** *There exists a critical probability $p_c = p_c^{(n)} \in (0,1)$ such that if $p < p_c$ then M is totally disconnected almost surely, and if $p > p_c$ then M contains a nontrivial connected component with positive probability.*

They actually prove more, the most powerful version states that in the supercritical case (i.e. when $p > p_c$) there is actually a unique unbounded component if the process is extended to the whole plane, but we will only concentrate on the most surprising fact that the critical probability is strictly between 0 and 1.

The main goal of the present section will be to prove the following generalization of the above theorem.

Theorem 2.34. *For every $d \in [0,2)$ there exists a critical probability $p_c^{(d)} = p_c^{(d,n)} \in (0,1)$ such that if $p < p_c^{(d)}$ then $\dim_{tH} M \leq d$ almost surely, and if $p > p_c^{(d)}$ then $\dim_{tH} M > d$ almost surely (provided $M \neq \emptyset$).*

In order to see that we actually obtain a generalization, just note that a compact space is totally disconnected iff $\dim_t M = 0$ ([12]), also that $\dim_{tH} M = 0$ iff $\dim_t M = 0$, and use $d = 0$. Theorem 2.33 basically says that certain curves show up at the critical probability, and our proof will show that even 'thick' families of curves show up, where the word thick is related to large Hausdorff dimension.

In the rest of this section first we do some preparations in the first subsection, then we prove the main theorem (Theorem 2.34) in the next subsection, and finally give an upper bound for $\dim_{tH} M$ and conclude that $\dim_{tH} M < \dim_H M$ almost surely in the non-trivial cases.

2.4.1 Preparation

For the proofs of the statements in the next two remarks see e.g. [10].

Remark 2.35. It is well-known from the theory of branching processes that $M = \emptyset$ almost surely iff $p \leq \frac{1}{n^2}$, so we may assume in the following that $p > \frac{1}{n^2}$.

If $\frac{1}{n^2} < p \leq \frac{1}{\sqrt{n}}$ then $\dim_t M = 0$ almost surely. Hence Fact 2.1 implies that $\dim_{tH} M = 0$ almost surely. (In fact, the same holds even for $p < p_c$, see Theorem 2.33.)

Remark 2.36. As for the Hausdorff dimension, for $p > \frac{1}{n^2}$ we have

$$\dim_H M = 2 + \frac{\log p}{\log n}$$

almost surely, provided $M \neq \emptyset$.

We will also need the 1-dimensional analogue of the process (intervals instead of squares). Here $M^{(1D)} = \emptyset$ almost surely iff $p \leq \frac{1}{n}$, and for $p > \frac{1}{n}$ we have

$$\dim_H M^{(1D)} = 1 + \frac{\log p}{\log n}$$

almost surely, provided $M^{(1D)} \neq \emptyset$.

Now we check that the almost sure topological Hausdorff dimension of M also exists.

Lemma 2.37. *For every $p > \frac{1}{n^2}$ and $n \geq 2$, $n \in \mathbb{N}$ there exists a number $d = d^{(p,n)} \in [0, 2]$ such that*

$$\dim_{tH} M = d$$

almost surely, provided $M \neq \emptyset$.

Proof. Let N be the random number of squares in M_1. Let us set $q = P(M = \emptyset)$. Then $q < 1$ by $p > \frac{1}{n^2}$, and [14, Thm. 15.2] gives that q is the least positive root of the polynomial

$$f(t) = -t + \sum_{k=0}^{n^2} P(N = k)t^k.$$

First we show that $P(\dim_{tH} M \leq x)$ is a root of f for every $x \in \mathbb{R}$.

Let $M_1 = \{Q_1, \dots, Q_N\}$, where the Q_i's are the first level subsquares, and fix $x \in \mathbb{R}$. (Define $Q_1 = \emptyset$ if $N = 0$.) For every i and k let $M_k^{Q_i}$ be the union of those squares in M_k that are in Q_i, and let $M^{Q_i} = \bigcap_k M_k^{Q_i}$. (Note that this is not the same as $M \cap Q_i$, since in this latter set there may be points on the boundary of Q_i 'coming from squares outside of Q_i'.) Then M^{Q_i} has the same distribution as a similar copy of M (this is called statistical self-similarity), and hence for every i

$$P\left(\dim_{tH} M^{Q_i} \leq x\right) = P\left(\dim_{tH} M \leq x\right).$$

Using the stability of the topological Hausdorff dimension for closed sets and the fact that the M^{Q_i}'s are independent and have the same distribution under the condition $N = k$, this implies

$$\begin{aligned}
P\left(\dim_{tH} M \leq x \,|\, N = k\right) &= P\left(\dim_{tH} M^{Q_i} \leq x \text{ for each } 1 \leq i \leq k \,|\, N = k\right) \\
&= \left(P\left(\dim_{tH} M^{Q_1} \leq x\right)\right)^k \\
&= \left(P\left(\dim_{tH} M \leq x\right)\right)^k.
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
P(\dim_{tH} M \leq x) &= \sum_{k=0}^{n^2} P(N = k) P(\dim_{tH} M \leq x \,|\, N = k) \\
&= \sum_{k=0}^{n^2} P(N = k)\left(P\left(\dim_{tH} M \leq x\right)\right)^k,
\end{aligned}$$

and thus $P(\dim_{tH} M \leq x)$ is indeed a root of f for every x.

As mentioned above, $q \neq 1$ and q is also a root of f. Moreover, 1 is obviously also a root, and it is easy to see that f is strictly convex on $(0, \infty)$, hence there are at most two positive roots. Hence q and 1 are the only roots, therefore $P(\dim_{tH} M \leq x) = q$ or 1 for every x.

Then the distribution function $F(x) = P(\dim_{tH} M \leq x \,|\, M \neq \emptyset)$ only attains the values 0 and 1, moreover, $F(0) = 0$, $F(2) = 1$, thus there is a value d where it 'jumps' from 0 to 1, and this concludes the proof. □

2.4.2 Proof of Theorem 2.34; the lower estimate of $\dim_{tH} M$

Set

$$p_c^{(d,n)} = \sup\left\{p : \dim_{tH} M^{(p,n)} \leq d \text{ almost surely}\right\}.$$

First we need some lemmas. The following one is analogous to [17, p. 387].

Lemma 2.38. *For every $d \in \mathbb{R}$ and $n \in \mathbb{N}$, $n \geq 2$*

$$p_c^{(d,n)} < 1 \quad \Longleftrightarrow \quad p_c^{(d,n^2)} < 1.$$

Proof. Clearly, it is enough to show that

$$p_c^{(d,n)} \left(1 - \left(1 - p_c^{(d,n)}\right)^{\frac{1}{n^2}}\right) \leq p_c^{(d,n^2)} \leq p_c^{(d,n)}. \tag{2.3}$$

We say that the random construction X is dominated by the random construction Y if they can be realized on the same probability space such that $X \subseteq Y$ almost surely.

Let us first prove the second inequality in (2.3). It clearly suffices to show that

$$\dim_{tH} M^{(p,n^2)} \leq d \text{ almost surely} \implies \dim_{tH} M^{(p,n)} \leq d \text{ almost surely}.$$

But this is rather straightforward, since $M_{2k}^{(p,n)}$ is easily seen to be dominated by $M_k^{(p,n^2)}$ for every k, hence $M^{(p,n)}$ is dominated by $M^{(p,n^2)}$.

Let us now prove the first inequality in (2.3). Set $\varphi(x) = 1 - (1-x)^{1/n^2}$. We need to show that

$$0 < p < p_c^{(d,n)} \varphi(p_c^{(d,n)}) \implies \dim_{tH} M^{(p,n^2)} \leq d \text{ almost surely}. \tag{2.4}$$

Since $x\varphi(x)$ is an increasing homeomorphism of the unit interval, $p = q\varphi(q)$ for some $q \in (0,1)$. Then clearly $q < p_c^{(d,n)}$, so $\dim_{tH} M^{(q,n)} \leq d$ almost surely. Therefore, in order to prove (2.4) it suffices to check that

$$M^{(p,n^2)} \text{ is dominated by } M^{(q,n)}. \tag{2.5}$$

First we check that

$$M_k^{(\varphi(q),n^2)} \text{ is dominated by } M_k^{(q,n)} \text{ for every } k, \tag{2.6}$$

and consequently $M^{(\varphi(q),n^2)}$ is dominated by $M^{(q,n)}$. Indeed, in the second case we erase a subsquare of side length $\frac{1}{n}$ with probability $1-q$ and keep it with probability q, while in the first case we *completely* erase a subsquare of side length $\frac{1}{n}$ with the same probability $(1-\varphi(q))^{n^2} = 1-q$ and hence keep *at least a subset of it* with probability q.

But this will easily imply (2.5), which will complete the proof. Indeed, after each step of the processes $M^{(\varphi(q),n^2)}$ and $M^{(q,n)}$ let us perform the following procedures. For $M^{(\varphi(q),n^2)}$ let us keep every existing square independently with probability q and erase it with probability $1-q$ (we do not do any subdivisions in this case). For $M^{(q,n)}$ let us take one more step of the construction of $M^{(q,n)}$. Using (2.6) this easily implies that $M_k^{(q\varphi(q),n^2)}$ is dominated by $M_{2k}^{(q,n)}$ for every k, hence $M^{(q\varphi(q),n^2)}$ is dominated by $M^{(q,n)}$, but $q\varphi(q) = p$, and hence (2.5) holds. □

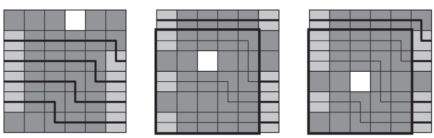
From now on let N be a fixed (large) positive integer to be chosen later. Recall that a square of level k is a set of the form $\left[\frac{i}{N^k}, \frac{i+1}{N^k}\right] \times \left[\frac{j}{N^k}, \frac{j+1}{N^k}\right] \subseteq [0,1]^2$.

Definition 2.39. A *walk of level* k is a sequence $(S_1, \dots, S_l)$ of non-overlapping squares of level k such that S_r and S_{r+1} are abutting for every $r = 1, \dots, l-1$, moreover $S_1 \cap (\{0\} \times [0,1]) \neq \emptyset$ and $S_l \cap (\{1\} \times [0,1]) \neq \emptyset$.

In particular, the only walk of level 0 is $([0,1]^2)$.

Definition 2.40. We say that $(S_1, \dots, S_l)$ is a *turning walk (of level* 1*)* if it satisfies the properties of a walk of level 1 except that instead of $S_l \cap (\{1\} \times [0,1]) \neq \emptyset$ we require that $S_l \cap ([0,1] \times \{1\}) \neq \emptyset$.

Lemma 2.41. *Let $\mathcal{S}$ be a set of $N-2$ distinct squares of level* 1 *intersecting* $\{0\} \times [0,1]$, *and let $\mathcal{T}$ be a set of $N-2$ distinct squares of level* 1 *intersecting* $\{1\} \times [0,1]$. *Moreover, let F^* be a square of level* 1 *such that the row of F^* does not intersect $\mathcal{S} \cup \mathcal{T}$. Then there exist $N-2$ non-overlapping walks of level* 1 *not containing F^* such that the set of their first squares coincides with $\mathcal{S}$ and the set of their last squares coincides with $\mathcal{T}$.*



Case 1. Case 2. Case 3.

Figure 2.1: Illustration to Lemma 2.41

Proof. The proof is by induction on N. The case $N = 2$ is obvious.

Case 1. *F^* is in the top or bottom row.*

By simply ignoring this row it is straightforward how to construct the walks in the remaining rows.

Case 2. *F^* is not in the top or bottom row, and both top corners or both bottom corners are in $\mathcal{S} \cup \mathcal{T}$.*

Without loss of generality we may suppose that both top corners are in $\mathcal{S} \cup \mathcal{T}$. Let the straight walk connecting these two corners be one of the walks to be constructed. Then let us shift the remaining members of $\mathcal{T}$ to the left by one square, and either we can apply the induction hypothesis to the $(N-1) \times (N-1)$ many squares in the bottom left corner of the original $N \times N$ many squares, or F^* is not among these $(N-1) \times (N-1)$ many squares and then the argument is even easier. Then one can see how to get the required walks.

Case 3. *Neither Case 1 nor Case 2 holds.*

Since there are only two squares missing on both sides, and F^* cannot be the top or bottom row, we infer that both $\mathcal{S}$ and $\mathcal{T}$ contain at least one corner. Since Case 2 does not hold, we obtain that both the top left and the bottom right corners or both the bottom left and the top right corners are in $\mathcal{S} \cup \mathcal{T}$. Without loss of generality we may suppose that both the top left and the bottom right corners are in $\mathcal{S} \cup \mathcal{T}$. By reflecting the picture about the center of the unit square if necessary, we may assume that F^* is not in the rightmost column. We now construct the first walk. Let it run straight from the top left corner to the top right corner, and then continue downwards until it first reaches a member of $\mathcal{T}$. Then, as above, we can similarly apply the induction hypothesis to the $(N-1) \times (N-1)$ many squares in the bottom left corner, and we are done. □

Lemma 2.42. *Let $\mathcal{S}$ be a set of $N-2$ distinct squares of level 1 intersecting $\{0\} \times [0,1]$, and let $\mathcal{T}$ be a set of $N-2$ distinct squares of level 1 intersecting $[0,1] \times \{1\}$ (the sets of starting and terminal squares). Moreover, let F^* be a square of level 1 (the forbidden square) such that the row of F^* does not intersect $\mathcal{S}$ and the column of F^* does not intersect $\mathcal{T}$. Then there exist $N-2$ non-overlapping turning walks not containing F^* such that the set of their first squares coincides with $\mathcal{S}$ and the set of their last squares coincides with $\mathcal{T}$.*

Proof. Obvious, just take the simplest 'L-shaped' walks. □

The last two lemmas will almost immediately imply the following.

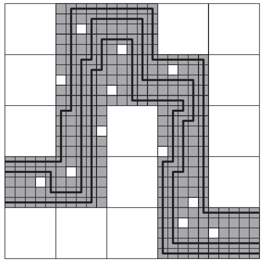


Figure 2.2: Illustration to Lemma 2.43

Lemma 2.43. *Let* $(S_1, \dots, S_l)$ *be a walk of level* k*, and* $\mathcal{F}$ *a system of squares of level* $k+1$ *such that each* S_r *contains at most* 1 *member of* $\mathcal{F}$*. Then* $(S_1, \dots, S_l)$ *contains* $N-2$ *non-overlapping subwalks of level* $k+1$ *avoiding* $\mathcal{F}$*.*

Proof. We may assume that each S_r contains exactly 1 member of $\mathcal{F}$. Let us denote the member of $\mathcal{F}$ in S_r by F_r^*. The subwalks will be constructed separately in each S_r, using an appropriately rotated or reflected version of either Lemma 2.42 or Lemma 2.41. It suffices to construct $\mathcal{S}_r$ and $\mathcal{T}_r$ for every r (compatible with F_r^*) so that for every member of $\mathcal{T}_r$ there is an abutting member of $\mathcal{S}_{r+1}$. (Of course we also have to make sure that every member of $\mathcal{S}_1$ intersects $\{0\} \times [0,1]$ and every member of $\mathcal{T}_l$ intersects $\{1\} \times [0,1]$.) For example, the construction of $\mathcal{T}_r$ for $r < l$ is as follows. The squares S_r and S_{r+1} share a common edge E. Assume for simplicity that E is horizontal. Then $\mathcal{T}_r$ will consist of those subsquares of S_r of level $k+1$ that intersect E and whose column differs from that of F_r^* and F_{r+1}^*. If these two columns happen to coincide then we can arbitrarily erase one more square. The remaining constructions are similar and the details are left to the reader. $\square$

Definition 2.44. We say that a square in M_k is 1-*full* if it contains at least N^2-1 many subsquares from M_{k+1}. We say that it is m-*full*, if it contains at least N^2-1 many $m-1$-full subsquares from M_{k+1}. We call M *full* if M_0 is m-full for every $m \in \mathbb{N}^+$.

The following lemma was the key realization in [10].

Lemma 2.45. *There exists a number* $p^{(N)} < 1$ *such that for every* $p > p^{(N)}$ *we have* $P\left(M^{(p,N)} \text{ is full}\right) > 0$*.*

See [10] or [14, Prop. 15.5] for the proof.

Definition 2.46. Let $L \leq N$ be positive integers. A compact set $K \subseteq [0,1]$ is called (L,N)-regular if it is of the form $K = \bigcap_{i\in\mathbb{N}} K_i$, where $K_0 = [0,1]$, and K_{k+1} is obtained by dividing every interval I in K_k into N many non-overlapping closed intervals of length $1/N^{k+1}$, and choosing L many of them for each I.

The following fact is well-known, see e.g. the more general [14, Thm. 9.3].

Fact 2.47. *An (L,N)-regular compact set has Hausdorff dimension $\frac{\log L}{\log N}$.*

Next we prove the main result of the present subsection.

Proof of Theorem 2.34. Let $d \in [0,2)$ be arbitrary. First we verify that, for sufficiently large N, if $M = M^{(p,N)}$ is full then $\dim_{tH} M > d$. The strategy is as follows. We define a collection $\mathcal{G}$ of disjoint connected subsets of M such that if a set intersects each member of $\mathcal{G}$ then its Hausdorff dimension is larger than $d-1$. Then we show that for every countable open basis $\mathcal{U}$ of M the union of the boundaries, $\bigcup_{U\in\mathcal{U}} \partial_M U$ intersects each member of $\mathcal{G}$, which clearly implies $\dim_{tH} M > d$.

Let us fix an integer N such that

$$N \geq 6 \text{ and } \frac{\log(N-2)}{\log N} > d-1, \tag{2.7}$$

and let us assume that M is full. Using Lemma 2.45 at each step we can choose $N-2$ non-overlapping walks of level 1 in M_1, then $N-2$ non-overlapping walks of level 2 in M_2 in each of the above walks, etc. Let us denote the obtained system at step k by

$$\mathcal{G}_k = \left\{\Gamma_{i_1,\dots,i_k} : (i_1,\dots,i_k) \in \{1,\dots,N-2\}^k\right\},$$

where $\Gamma_{i_1,\dots,i_k}$ is the union of the squares of the corresponding walk. (Set $\mathcal{G}_0 = \{\Gamma_\emptyset\} = \{[0,1]^2\}$.) Let us also put

$$C_k = \left\{y \in [0,1] : (0,y) \in \bigcup \mathcal{G}_k\right\}$$

and define

$$C = \bigcap_{k\in\mathbb{N}} C_k.$$

Then clearly C is an $(N-2,N)$-regular compact set, therefore Fact 2.47 yields that $\dim_H C = \frac{\log(N-2)}{\log N} > d-1$. We will also need that $\dim_H(C \setminus \mathbb{Q}) > d-1$, but this is clear since $\dim_H C > 0$ and hence $\dim_H(C \setminus \mathbb{Q}) = \dim_H C$.

For every $y \in C \setminus \mathbb{Q}$ and every $k \in \mathbb{N}$ there is a unique $(i_1, \dots, i_k)$ such that $(0, y) \in \Gamma_{i_1,\dots,i_k}$. (For y's of the form $\frac{i}{N^l}$ there may be two such $(i_1, \dots, i_k)$'s, and we would like to avoid this complication.) Put $\Gamma_k(y) = \Gamma_{i_1,\dots,i_k}$ and $\Gamma(y) = \bigcap_{k=1}^{\infty} \Gamma_k(y)$. Since $\Gamma(y)$ is a decreasing intersection of compact connected sets, it is itself connected ([12]). (Actually, it is a continuous curve, but we will not need this here.) It is also easy to see that it intersects $\{0\} \times [0, 1]$ and $\{1\} \times [0, 1]$.

We can now define

$$\mathcal{G} = \{\Gamma(y) : y \in C \setminus \mathbb{Q}\}.$$

Next we prove that $\mathcal{G}$ consists of disjoint sets. Let $y, y' \in C \setminus \mathbb{Q}$ be distinct. Pick $l \in \mathbb{N}$ so large such that $|y - y'| > \frac{6}{N^l}$. Then there are at least 5 intervals of level l between y and y'. Since we always chose $N-2$ intervals out of N along the construction, there can be at most 4 consecutive non-selected intervals, therefore there is a $\Gamma_{i_1,\dots,i_l}$ separating y and y'. But then this also separates $\Gamma_l(y)$ and $\Gamma_l(y')$, hence $\Gamma(y)$ and $\Gamma(y')$ are disjoint.

Now we check that for every $y \in C \setminus \mathbb{Q}$ and every countable open basis $\mathcal{U}$ of M the set $\bigcup_{U \in \mathcal{U}} \partial_M U$ intersects $\Gamma(y)$. Let $z_0 \in \Gamma(y)$ and $U_0 \in \mathcal{U}$ such that $z_0 \in U_0$ and $\Gamma(y) \nsubseteq U_0$. Then $\partial_M U_0$ must intersect $\Gamma(y)$, since otherwise $\Gamma(y) = (\Gamma(y) \cap U_0) \cup (\Gamma(y) \cap \operatorname{int}_M(M \setminus U_0))$, hence a connected set would be the union of two non-empty disjoint relatively open sets, a contradiction.

Thus, as explained in the first paragraph of the proof, it is sufficient to prove that if a set Z intersects every $\Gamma(y)$ then $\dim_H Z > d - 1$. This is easily seen to hold if we can construct an onto Lipschitz map

$$\varphi \colon \bigcup \mathcal{G} \to C \setminus \mathbb{Q}$$

that is constant on every member of $\mathcal{G}$, since Lipschitz maps do not increase Hausdorff dimension, and $\dim_H(C \setminus \mathbb{Q}) > d - 1$. Define

$$\varphi(z) = y \text{ if } z \in \Gamma(y),$$

which is well-defined by the disjointness of the members of $\mathcal{G}$.

Let us now prove that this map is Lipschitz. Let $y, y' \in C \setminus \mathbb{Q}$, $z \in \Gamma(y)$, and $z' \in \Gamma(y')$. Choose $l \in \mathbb{N}^+$ such that $\frac{1}{N^l} < |y - y'| \le \frac{1}{N^{l-1}}$. Then using $N \ge 6$ we obtain $|y - y'| > \frac{6}{N^{l+1}}$, thus, as above, there is a walk of level $l+1$ separating z and z'. Therefore $|z - z'| \ge \frac{1}{N^{l+1}}$, and hence

$$|\varphi(z) - \varphi(z')| = |y - y'| \le \frac{1}{N^{l-1}} = N^2 \frac{1}{N^{l+1}} \le N^2 |z - z'|,$$

therefore φ is Lipschitz with Lipschitz constant at most N^2.

To finish the proof, let n be given as in Theorem 2.34 and pick $k \in \mathbb{N}$ so large that $N = n^{2^k}$ satisfies (2.7). If $p > p^{(N)}$ then using Lemma 2.45 we deduce that

$$P\left(\dim_{tH} M^{(p,N)} > d\right) \geq P\left(M^{(p,N)} \text{ is full }\right) > 0,$$

which implies $p_c^{(d,N)} < 1$. Iterating k times Lemma 2.38 we infer $p_c^{(d,n)} < 1$.

Now, if $p > p_c^{(d,n)}$ then

$$P\left(\dim_{tH} M^{(p,n)} > d \,\middle|\, M^{(p,n)} \neq \emptyset\right) \geq P(\dim_{tH} M^{(p,n)} > d) > 0.$$

Combining this with Lemma 2.37 we deduce that

$$P\left(\dim_{tH} M^{(p,n)} > d \,\middle|\, M^{(p,n)} \neq \emptyset\right) = 1,$$

which completes the proof of the theorem. □

Remark 2.48. It is well-known and not difficult to see that $\lim\limits_{p\to 1} P(M^{(p,n)} = \emptyset) = 0$. Using this it is an easy consequence of the previous theorem that for every integer $n > 1$, $d < 2$ and $\varepsilon > 0$ there exists a $\delta = \delta^{(n,d,\varepsilon)} > 0$ such that for all $p > 1 - \delta$

$$P\left(\dim_{tH} M^{(p,n)} > d\right) > 1 - \varepsilon.$$

2.4.3 The upper estimate of $\dim_{tH} M$

The argument of this subsection will rely on some ideas from [10].

Theorem 2.49. *If $p > \frac{1}{\sqrt{n}}$ then almost surely*

$$\dim_{tH} M \leq 2 + 2\frac{\log p}{\log n}.$$

Proof. A segment is called a *basic segment* if it is of the form $\left[\frac{i-1}{n^k}, \frac{i}{n^k}\right] \times \left\{\frac{j}{n^k}\right\}$ or $\left\{\frac{j}{n^k}\right\} \times \left[\frac{i-1}{n^k}, \frac{i}{n^k}\right]$, where $k \in \mathbb{N}^+$, $i \in \{1, ..., n^k\}$ and $j \in \{1, ..., n^k - 1\}$.

It suffices to show that for every basic segment S and for every $\varepsilon > 0$ there exists (almost surely, a random) arc $\gamma \subseteq [0,1]^2$ connecting the endpoints of S in the ε-neighborhood of S such that $\dim_H (M \cap \gamma) \leq 1 + 2\frac{\log p}{\log n}$. Indeed, we can almost surely construct the analogous arcs for all basic segments, and hence obtain a basis of M consisting of 'approximate squares' whose boundaries are of Hausdorff dimension at most $1 + 2\frac{\log p}{\log n}$, therefore $\dim_{tH} M \leq 2 + 2\frac{\log p}{\log n}$ almost surely.

Let us now construct such an arc γ for S and $\varepsilon > 0$. We may assume that S is horizontal, hence it is of the form $S = \left[\frac{i-1}{n^k}, \frac{i}{n^k}\right] \times \left\{\frac{j}{n^k}\right\}$ for some $k \in \mathbb{N}^+$, $i \in \{1, ..., n^k\}$ and $j \in \{1, ..., n^k - 1\}$.

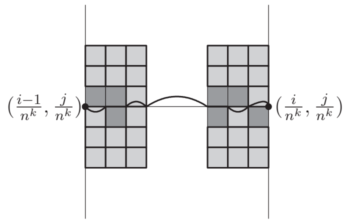


Figure 2.3: Construction of the arc γ connecting the endpoints of S

We divide S into n subsegments of length $\frac{1}{n^{k+1}}$, and a subsegment $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times \{\frac{j}{n^k}\}$ is called *bad* if both the adjacent squares $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times [\frac{j}{n^k} - \frac{1}{n^{k+1}}, \frac{j}{n^k}]$ and $\left[\frac{m-1}{n^{k+1}}, \frac{m}{n^{k+1}}\right] \times [\frac{j}{n^k}, \frac{j}{n^k} + \frac{1}{n^{k+1}}]$ are in M_{k+1}. Otherwise we say that the subsegment is *good*. Let B_1 denote the union of the bad segments. Then inside every bad segment we repeat the same procedure, and obtain B_2 and so on. It is easy to see that this process is (a scaled copy of) the 1-dimensional fractal percolation with p replaced by p^2. Let $B = \bigcap_l B_l$ be its limit set. Then by Remark 2.36 (note that $p^2 > \frac{1}{n}$) we obtain $\dim_H B = 1 + \frac{\log p^2}{\log n} = 1 + 2\frac{\log p}{\log n}$ or $B = \emptyset$ almost surely. So it suffices to construct a γ connecting the endpoints of S in the ε-neighborhood of S such that $\gamma \cap M = B$ (except perhaps some endpoints, but all the endpoints form a countable set, hence a set of Hausdorff dimension 0).

But this is easily done. Indeed, for every good subsegment I let γ_I be an arc connecting the endpoints of I in a small neighborhood of I such that γ is disjoint from M apart from the endpoints (this is possible, since either the top or the bottom square was erased from M). Then $\gamma = \left(\bigcup_{I \text{ is good}} \gamma_I\right) \cup B$ works. $\square$

Using Remarks 2.35 and 2.36 this easily implies

Corollary 2.50. *Almost surely*

$$\dim_{tH} M < \dim_H M \text{ or } M = \emptyset.$$

Remark 2.51. Calculating the exact value of $\dim_{tH} M$ seems to be difficult, since it would provide the value of the critical probability p_c of Chayes, Chayes and Durrett (where the phase transition occurs, see above), and this is a long-standing open problem.

2.5 Application II: The Hausdorff dimension of the level sets of the generic continuous function

Now we return to Problem 1.3. The main goal is to find analogues to Kirchheim's theorem, that is, to determine the Hausdorff dimension of the level sets of the generic continuous function defined on a compact metric space K.

Let us first note that the case $\dim_t K = 0$, that is, when there is a basis consisting of clopen sets is trivial because of the following well-known and easy fact.

Fact 2.52. *If K is a compact metric space with $\dim_t K = 0$ then the generic continuous function is one-to-one on K.*

Corollary 2.53. *If K is a compact metric space with $\dim_t K = 0$ then every non-empty level set of the generic continuous function is of Hausdorff dimension 0.*

Hence from now on we can restrict our attention to the case of positive topological dimension.

In the first part of this section we prove Theorem 2.71 and Corollary 2.72, our main theorems concerning level sets of the generic function defined on an arbitrary compact metric space, then we use this to derive conclusions about homogeneous and self-similar spaces in Theorem 2.73 and Corollary 2.74.

2.5.1 Arbitrary compact metric spaces

The goal of this subsection is to prove Theorem 2.71. In order to do this we will need a sequence of equivalent definitions of the topological Hausdorff dimension. These equivalent definitions may be of some interest in their own right.

Let us fix a compact metric space K with $\dim_t K > 0$, and let $C(K)$ denote the space of continuous real-valued functions equipped with the supremum norm. Since this is a complete metric space, we can use Baire category arguments.

Definition 2.54. We say that a continuous function is *d-level narrow*, if there exists a dense set $S_f \subseteq \mathbb{R}$ such that $\dim_H f^{-1}(y) \leq d-1$ for every $y \in S_f$. Let $\mathcal{N}_d$ be the set of d-level narrow functions. Define

$$P_n = \{d : \mathcal{N}_d \text{ is somewhere dense in } C(K)\},$$

and let $\dim_n K = \inf P_n$.

We repeat the definition of the topological Hausdorff dimension in an analogous form.

Definition 2.55. Define

$$P_{tH} = \{d : K \text{ has a basis } \mathcal{U} \text{ such that } \dim_H \partial U \leq d-1 \text{ for every } U \in \mathcal{U}\},$$

then $\dim_{tH} K = \inf P_{tH}$.

For the next definition we need the following notation.

Notation 2.56. If $\mathcal{A}$ is a family of sets then let $T(\mathcal{A})$ denote the set of points covered by at least two members of $\mathcal{A}$, that is, $T(\mathcal{A}) = \bigcup_{A_1, A_2 \in \mathcal{A}, A_1 \neq A_2} (A_1 \cap A_2)$.

Definition 2.57. We say that $\mathcal{C}$ is a *d-dimensional small cover for $\varepsilon > 0$*, if $\mathcal{C}$ is a finite family of compact sets such that $\bigcup \mathcal{C} = K$, $\operatorname{diam} C \leq \varepsilon$ for all $C \in \mathcal{C}$ and $\dim_H T(\mathcal{C}) \leq d-1$. Define

$$P_s = \{d : \forall \varepsilon > 0, \ \exists \text{ a } d\text{-dimensional small cover for } \varepsilon\},$$

and let $\dim_s K = \inf P_s$.

Definition 2.58. For $d \geq 1$ we say that $\mathcal{C}$ is a a *d-dimensional pre-measure fat packing for $\varepsilon > 0$* if $\mathcal{C}$ is a finite family of disjoint compact subsets of K such that $\operatorname{diam} C \leq \varepsilon$ for all $C \in \mathcal{C}$ and $\mathcal{H}^{d-1+\varepsilon}_{\infty}(K \setminus \bigcup \mathcal{C}) \leq \varepsilon$. Define

$$P_m = \{d \geq 1 : \forall \varepsilon > 0, \ \exists \text{ a } d\text{-dimensional pre-measure fat packing for } \varepsilon\},$$

and let $\dim_m K = \inf P_m$.

Definition 2.59. Define

$$P_l = \left\{d : \text{for the generic } f \in C(K), \forall y \in \mathbb{R} \ \dim_H f^{-1}(y) \leq d-1\right\},$$

and let $\dim_l K = \inf P_l$ be the *generic level set dimension* of K.

We assume that by definition $\infty \in P_n, P_{tH}, P_s, P_m, P_l$.

Our goal is to show in Theorem 2.62 that if $\dim_t K > 0$ then our five notions of dimension coincide. One can verify that they may differ if $\dim_t K = 0$.

Remark 2.60. The restriction $d \geq 1$ in Definition 2.58 is not too artificial. It is easy to check directly that if $\dim_t K > 0$ then $P_n, P_{tH}, P_s, P_m, P_l \subseteq [1, \infty]$. However, it will also be a consequence of Fact 2.2 and Theorem 2.62.

First we need a technical lemma related to Definition 2.58.

Lemma 2.61. *Let X be a metric space, $0 \le c < \infty$ and $c_n \searrow c$. If $\mathcal{H}^{c_n}_\infty(X) \to 0$ then $\dim_H X \le c$.*

Proof. We may assume that $\mathcal{H}^{c_n}_\infty(X) < 1$. Fix $t > c$. Then $c_n < t$ for large enough n. It is not difficult to see that $c_n < t$ and $\mathcal{H}^{c_n}_\infty(X) < 1$ imply $\mathcal{H}^{t}_\infty(X) \le \mathcal{H}^{c_n}_\infty(X)$. Therefore $\mathcal{H}^{c_n}_\infty(X) \to 0$ yields $\mathcal{H}^{t}_\infty(X) = 0$, which easily implies $\mathcal{H}^{t}(X) = 0$. Hence $\dim_H X \le t$, and since $t > c$ was arbitrary, $\dim_H X \le c$. $\square$

Theorem 2.62. *If K is a compact metric space with $\dim_t K > 0$ then $P_n = P_{tH} = P_s = P_m = P_l$.*

This immediately yields.

Corollary 2.63. *If K is a compact metric space with $\dim_t K > 0$ then $\dim_n K = \dim_{tH} K = \dim_s K = \dim_m K = \dim_l K$.*

This result will be a technical tool in the sequel, but we believe that the equation $\dim_{tH} K = \dim_s K$ is particularly interesting in its own right. Let us reformulate it now.

Corollary 2.64. *If K is a compact metric space then $\dim_{tH} K$ is the smallest number d for which K can be covered by a finite family of compact sets of arbitrarily small diameter such that the set of points covered more than once has Hausdorff dimension at most $d-1$.*

Next we prove Theorem 2.62. The proof will consist of five lemmas.

Lemma 2.65. *$P_n \subseteq P_{tH}$.*

Proof. Assume $d \in P_n$ and $d < \infty$. Let us fix $x_0 \in K$ and $r > 0$. To verify $d \in P_{tH}$ we need to find an open set U such that $x_0 \in U \subseteq U(x_0, r)$ and $\dim_H \partial U \le d-1$. We may assume $\partial U(x_0, r) \ne \emptyset$, otherwise we are done.

By $d \in P_n$ we obtain that $\mathcal{N}_d$ is dense in a ball $B(f_0, 6\varepsilon)$, $\varepsilon > 0$. By decreasing r if necessary, we may assume that $\operatorname{diam}(f_0(U(x_0,r))) \le 3\varepsilon$. Then Tietze's Extension Theorem provides an $f \in B(f_0, 6\varepsilon)$ such that $f(x_0) = f_0(x_0)$ and $f|_{\partial U(x_0,r)}(x) = f_0(x_0) + 3\varepsilon$ for every $x \in \partial U(x_0, r)$. Since $\mathcal{N}_d$ is dense in $B(f_0, 6\varepsilon)$, we can choose $g \in \mathcal{N}_d$ such that $||f - g|| \le \varepsilon$. By the construction of g it follows that $g(x_0) < \min\{g(\partial U(x_0, r))\}$. Hence in the dense set S_g (see Definition 2.54) there is an $s \in S_g$ such that

$$g(x_0) < s < \min\{g(\partial U(x_0, r))\}. \tag{2.8}$$

Let

$$U = g^{-1}\left((-\infty, s)\right) \cap U(x_0, r),$$

then clearly $x_0 \in U \subseteq U(x_0, r)$. By (2.8) we have $\partial g^{-1}\left((-\infty, s)\right) \cap \partial U(x_0, r) = \emptyset$, therefore $\partial U \subseteq \partial g^{-1}\left((-\infty, s)\right) \subseteq g^{-1}(s)$. Using $s \in S_g$ we infer that $\dim_H \partial U \leq \dim_H g^{-1}(s) \leq d - 1$. □

Lemma 2.66. $P_{tH} \subseteq P_s$.

Proof. Assume $d \in P_{tH}$ and $d < \infty$. Fix $\varepsilon > 0$, and let $\mathcal{U}$ be an open basis of K such that $\dim_H \partial U \leq d - 1$ for all $U \in \mathcal{U}$. Then $\{U \in \mathcal{U} : \operatorname{diam} U \leq \varepsilon\}$ covers K, hence by compactness there exists a finite subcover $\{U_i\}_{i=1}^k$. Then $\operatorname{diam} U_i \leq \varepsilon$ and $\dim_H \partial U_i \leq d - 1$ for all i. Let us now consider all the 2^k possible sets of the form $U_1^{\alpha_1} \cap \dots \cap U_k^{\alpha_k}$, where every $\alpha_i \in \{1, -1\}$, and $U_i^1 = \operatorname{cl} U_i$, and $U_i^{-1} = K \setminus U_i$. Let $\mathcal{C}$ be the family consisting of the sets of the above form, then it is easy to check that $\mathcal{C}$ covers K and also that $\operatorname{diam} C \leq \varepsilon$ for every $C \in \mathcal{C}$ (note that $\operatorname{diam}(U_1^{-1} \cap \dots \cap U_k^{-1}) \leq \varepsilon$ holds simply because $U_1^{-1} \cap \dots \cap U_k^{-1} = \emptyset$). Moreover, one can check that $T(\mathcal{C}) \subseteq \bigcup_{i=1}^k \partial U_i$, hence $\dim_H T(\mathcal{C}) \leq \dim_H \bigcup_{i=1}^k \partial U_i \leq d - 1$. Therefore, $\mathcal{C}$ is a d-dimensional small cover for $\varepsilon > 0$, hence $d \in P_s$. □

Lemma 2.67. $P_s \subseteq P_m$.

Proof. Assume $d \in P_s$ and $d < \infty$. First we check that $\dim_t K > 0$ implies $d \geq 1$. If $d < 1$ and for every $\varepsilon > 0$ there exists a d-dimensional small cover for ε then these covers are actually finite partitions (since $d - 1 < 0$) into compact sets, but then these sets are also open, hence K has a clopen basis, which is impossible.

Fix $\varepsilon > 0$. Let $\mathcal{C}$ be a d-dimensional small cover for ε. Since $\dim_H T(\mathcal{C}) \leq d - 1$, we can choose an open set V such that $T(\mathcal{C}) \subseteq V$ and $\mathcal{H}_\infty^{d-1+\varepsilon}(V) \leq \varepsilon$. Let $\mathcal{C}' = \{C \setminus V : C \in \mathcal{C}\}$, then $\mathcal{C}'$ is a finite family of disjoint compact sets. Clearly, $\operatorname{diam} C' \leq \varepsilon$ for every $C' \in \mathcal{C}'$. Since $K \setminus \bigcup \mathcal{C}' = V$, we also have $\mathcal{H}_\infty^{d-1+\varepsilon}(K \setminus \bigcup \mathcal{C}') \leq \varepsilon$, hence $\mathcal{C}'$ is a d-dimensional pre-measure fat packing for ε and thus $d \in P_m$. □

Lemma 2.68. $P_m \subseteq P_l$.

Proof. Assume $d \in P_m$ and $d < \infty$. By definition, $d \geq 1$. First assume $d > 1$. For $n \in \mathbb{N}^+$ let $\mathcal{C}^n = \{C_1^n, \dots, C_{k_n}^n\}$ be a d-dimensional pre-measure fat packing for $1/n$, that is, $\mathcal{C}^n$ consists of disjoint compact sets, for all $i \in \{1, \dots, k_n\}$

$$\operatorname{diam} C_i^n \leq \frac{1}{n}, \tag{2.9}$$

and for the open set $V^n = K \setminus \bigcup_{i=1}^{k_n} C_i^n$ we have

$$\mathcal{H}_\infty^{d-1+\frac{1}{n}}(V^n) = \mathcal{H}_\infty^{d-1+\frac{1}{n}}\left(K \setminus \bigcup \mathcal{C}^n\right) \leq \frac{1}{n}. \tag{2.10}$$

Let

$$\mathcal{R}(n) = \left\{(r_1, \dots, r_{k_n}) \in \mathbb{Q}^{k_n} : r_i \neq r_j \text{ if } i \neq j\right\},$$

and let $f_{n,r_1,\dots,r_{k_n}} \in C(K)$ be a continuous function that is constant r_i on C_i^n for every $i \in \{1, \dots, k_n\}$. It is not difficult to see (using that every element of $C(K)$ is uniformly continuous) that for every $N \in \mathbb{N}^+$ the set

$$\{f_{n,r_1,\dots,r_{k_n}} : n \geq N,\ (r_1, \dots, r_{k_n}) \in \mathcal{R}(n)\}$$

is dense in $C(K)$.

Set

$$\delta_{r_1,\dots,r_{k_n}} = \min\left\{\frac{1}{n}, \frac{|r_i - r_j|}{3} : i, j \in \{1, \dots, k_n\},\ i \neq j\right\} > 0,$$

then

$$\mathcal{G}(N) = \bigcup_{n \geq N} \bigcup_{(r_1,\dots,r_{k_n}) \in \mathcal{R}(n)} U(f_{n,r_1,\dots,r_{k_n}}, \delta_{r_1,\dots,r_{k_n}})$$

is dense open in $C(K)$ for all $N \in \mathbb{N}^+$. Therefore

$$\mathcal{G} = \bigcap_{N \in \mathbb{N}^+} \mathcal{G}(N)$$

is co-meager in $C(K)$.

It remains to prove that for all $f \in \mathcal{G}$ all level sets of f are of Hausdorff dimension at most $d-1$. Fix $f \in \mathcal{G}$ and $y \in \mathbb{R}$. By the definition of $\mathcal{G}$, there are infinitely many $n \in \mathbb{N}^+$ such that f is in one of the $U(f_{n,r_1,\dots,r_{k_n}}, \delta_{r_1,\dots,r_{k_n}})$'s. For every such n there exists $i = i(y, n)$ such that

$$f^{-1}(y) \subseteq C_i^n \cup V^n.$$

Using (2.9), (2.10) (C_i^n is covered by itself when estimating $\mathcal{H}_\infty^{d-1+\frac{1}{n}}(C_i^n)$) and finally $d > 1$ we obtain

$$\mathcal{H}_\infty^{d-1+\frac{1}{n}}\left(f^{-1}(y)\right) \leq \mathcal{H}_\infty^{d-1+\frac{1}{n}}(C_i^n) + \mathcal{H}_\infty^{d-1+\frac{1}{n}}(V^n) \leq \tag{2.11}$$

$$\left(\frac{1}{n}\right)^{d-1+\frac{1}{n}} + \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

Let us now fix a sequence n_k of integers for which (2.11) holds. By applying Lemma 2.61 for $c_k = d - 1 + \frac{1}{n_k} \searrow d - 1$ we obtain that for all $f \in \mathcal{G}$ and $y \in \mathbb{R}$

$$\dim_H f^{-1}(y) \leq d - 1.$$

Therefore, $d \in P_l$.

Let us now consider $d = 1$. Fix a sequence $d_n \searrow 1$. By the previous case we have $d_n \in P_l$ for all $n \in \mathbb{N}$, hence there exist co-meager sets $\mathcal{F}_n \subseteq C(K)$ such that $\dim_H f^{-1}(y) \le d_n - 1$ for all $f \in \mathcal{F}_n$ and $y \in \mathbb{R}$. The set $\mathcal{F} = \bigcap_{n\in\mathbb{N}} \mathcal{F}_n$ is co-meager in $C(K)$, and obviously $\dim_H f^{-1}(y) \le \lim_{n\to\infty}(d_n - 1) = 0$ for all $f \in \mathcal{F}$ and $y \in \mathbb{R}$. Thus $1 \in P_l$. $\square$

Lemma 2.69. $P_l \subseteq P_n$.

Proof. Assume $d \in P_l$ and $d < \infty$. By the definition of P_l, for the generic continuous function f we have $\dim_H f^{-1}(y) \le d - 1$ for all $y \in \mathbb{R}$. Hence $\mathcal{N}_d$ is co-meager, thus (everywhere) dense. Hence $d \in P_n$. $\square$

This concludes the proof of Theorem 2.62.

For the proof of the main theorem of this section we will need that the infima in the above definitions are actually attained.

Lemma 2.70. *If K is a compact metric space with $\dim_t K > 0$ then $\dim_n K = \min P_n$, $\dim_{tH} K = \min P_{tH}$, $\dim_s K = \min P_s$, $\dim_m K = \min P_m$, and $\dim_l K = \min P_l$.*

Proof. By Theorem 2.62 it suffices to prove that P_m has a minimal element. We may assume $P_m \neq \{\infty\}$. Let $d = \inf P_m$. Fix $\varepsilon > 0$. There exists $d' \in P_m$ such that $d' \ge d$ and $d' - d < \varepsilon$. Set $\varepsilon' = \varepsilon - (d' - d)$, then $0 < \varepsilon' \le \varepsilon$ and $d' - 1 + \varepsilon' = d - 1 + \varepsilon$. By $d' \in P_m$ there exists a d'-dimensional pre-measure fat packing $\mathcal{C}$ for ε', that is, a finite disjoint family of compact sets such that $\operatorname{diam} C \le \varepsilon'$ for all $C \in \mathcal{C}$ and $\mathcal{H}^{d'-1+\varepsilon'}_\infty(K \setminus \bigcup \mathcal{C}) \le \varepsilon'$. But then by $\varepsilon' \le \varepsilon$ and $d' - 1 + \varepsilon' = d - 1 + \varepsilon$ we obtain that $\mathcal{C}$ is also d-dimensional pre-measure fat packing for ε, and hence $d \in P_m$. $\square$

Now we are ready to describe the Hausdorff dimension of the level sets of generic continuous functions.

As already mentioned above, if $\dim_t K = 0$ then every level set of a generic continuous function on K consists of at most one point.

Theorem 2.71. *If K is a compact metric space with $\dim_t K > 0$ then for the generic $f \in C(K)$*

(i) $\dim_H f^{-1}(y) \le \dim_{tH} K - 1$ *for every* $y \in \mathbb{R}$,

(ii) for every $\varepsilon > 0$ there exists a non-degenerate interval $I_{f,\varepsilon}$ such that $\dim_H f^{-1}(y) \ge \dim_{tH} K - 1 - \varepsilon$ for every $y \in I_{f,\varepsilon}$.

Proof. By Lemma 2.70 we have $\dim_l K = \min P_l$ and hence $\dim_l K \in P_l$. By the definition of P_l and using Corollary 2.63 we deduce that there is a co-meager set $\mathcal{F} \subseteq C(K)$ such that for every $f \in \mathcal{F}$ and $y \in \mathbb{R}$

$$\dim_H f^{-1}(y) \leq \dim_l K - 1 = \dim_{tH} K - 1,$$

therefore (i) holds.

Let us now prove (ii). Clearly, by Theorem 2.62, $\dim_{tH} K - \frac{1}{k} < \dim_{tH} K = \dim_n K$ for every $k \in \mathbb{N}^+$. Hence $\mathcal{N}_{\dim_{tH} K - \frac{1}{k}}$ is nowhere dense by the definition of $\dim_n K$. It follows from the definition of $\mathcal{N}_d$ that for every $f \in C(K) \setminus \mathcal{N}_{\dim_{tH} K - \frac{1}{k}}$ there exists a non-trivial interval $I_{f,\frac{1}{k}}$ such that $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \frac{1}{k}$ for every $y \in I_{f,\frac{1}{k}}$. But then (ii) holds for every $f \in C(K) \setminus (\bigcup_{k \in \mathbb{N}^+} \mathcal{N}_{\dim_{tH} K - \frac{1}{k}})$, and this latter set is clearly co-meager, which concludes the proof of the theorem. □

This immediately implies

Corollary 2.72. *If K is a compact metric space with $\dim_t K > 0$ then we have $\sup\{\dim_H f^{-1}(y) : y \in \mathbb{R}\} = \dim_{tH} K - 1$ for the generic $f \in C(K)$.*

2.5.2 Homogeneous and self-similar compact metric spaces

In this subsection we show that if the compact metric space is sufficiently homogeneous, e.g. self-similar (see [14] or [29]) then we can say much more.

Theorem 2.73. *Let K be a compact metric space with $\dim_t K > 0$, and assume that $\dim_{tH} B(x,r) = \dim_{tH} K$ for every $x \in K$ and $r > 0$. Then for the generic $f \in C(K)$ for the generic $y \in f(K)$ we have*

$$\dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

Before turning to the proof of this theorem we formulate a corollary. Recall that K is self-similar if there are contractive similitudes $\varphi_1, \dots, \varphi_k : K \to K$ such that $K = \bigcup_{i=1}^k \varphi_i(K)$. The sets of the form $\varphi_{i_1} \circ \varphi_{i_2} \circ \dots \varphi_{i_m}(K)$ are called the elementary pieces of K. It is easy to see that every ball in K contains an elementary piece. Moreover, by Corollary 2.8 the topological Hausdorff dimension of every elementary piece is $\dim_{tH} K$. Hence, using monotonicity as well, we obtain that if K is self-similar then $\dim_{tH} B(x,r) = \dim_{tH} K$ for every $x \in K$ and $r > 0$. This yields the following.

Corollary 2.74. *Let K be a self-similar compact metric space with $\dim_t K > 0$. Then for the generic $f \in C(K)$ for the generic $y \in f(K)$ we have*

$$\dim_H f^{-1}(y) = \dim_{tH} K - 1.$$

Before proving Theorem 2.73 we need a lemma.

Lemma 2.75. *Let $K_1 \subseteq K_2$ be compact metric spaces and*

$$R\colon C(K_2) \to C(K_1), \quad R(f) = f|_{K_1}.$$

If $\mathcal{F} \subseteq C(K_1)$ is co-meager then so is $R^{-1}(\mathcal{F}) \subseteq C(K_2)$.

Proof. The map R is clearly continuous. Using the Tietze Extension Theorem it is not difficult to see that it is also open. We may assume that $\mathcal{F}$ is a dense G_δ set in $C(K_1)$. The continuity of R implies that $R^{-1}(\mathcal{F})$ is also G_δ, thus it is enough to prove that $R^{-1}(\mathcal{F})$ is dense in $C(K_2)$. Let $\mathcal{U} \subseteq C(K_2)$ be non-empty open, then $R(\mathcal{U}) \subseteq C(K_1)$ is also non-empty open, hence $R(\mathcal{U}) \cap \mathcal{F} \neq \emptyset$, and therefore $\mathcal{U} \cap R^{-1}(\mathcal{F}) \neq \emptyset$. □

Proof of Theorem 2.73. Theorem 2.71 implies that for the generic $f \in C(K)$ for *every* $y \in \mathbb{R}$ we have $\dim_H f^{-1}(y) \leq \dim_{tH} K - 1$, so we only have to prove the opposite inequality.

For $f \in C(K)$ and $\varepsilon > 0$ let

$$L_{f,\varepsilon} = \{y \in f(K) : \dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon\}.$$

First we show that it suffices to construct for every $\varepsilon \in (0,1)$ a co-meager set $\mathcal{F}_\varepsilon \subseteq C(K)$ such that for every $f \in \mathcal{F}_\varepsilon$ the set $L_{f,\varepsilon}$ is co-meager in $f(K)$. Indeed, then the set $\mathcal{F} = \bigcap_{k\in\mathbb{N},k\geq 2} \mathcal{F}_{\frac{1}{k}} \subseteq C(K)$ is co-meager, and for every $f \in \mathcal{F}$ the set $L_f = \bigcap_{k\in\mathbb{N},k\geq 2} L_{f,\frac{1}{k}} \subseteq f(K)$ is also co-meager. Since for every $y \in L_f$ clearly $\dim_H f^{-1}(y) \geq \dim_{tH} K - 1$, this finishes the proof.

Let us now construct such an $\mathcal{F}_\varepsilon$ for a fixed $\varepsilon \in (0,1)$. Let $\{B_n\}_{n\in\mathbb{N}}$ be a countable basis of K consisting of closed balls, and for all $n \in \mathbb{N}$ let $R_n\colon C(K) \to C(B_n)$ be defined as

$$R_n(f) = f|_{B_n}.$$

Let us also define

$$\mathcal{B}_n = \left\{f \in C(B_n) : \exists I_{f,\varepsilon} \text{ s. t. } \forall y \in I_{f,\varepsilon} \ \dim_H f^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon\right\},$$

(where $I_{f,\varepsilon}$ is understood to be a non-degenerate interval). Finally, let us define

$$\mathcal{F}_\varepsilon = \bigcap_{n\in\mathbb{N}} R_n^{-1}(\mathcal{B}_n).$$

First we show that $\mathcal{F}_\varepsilon$ is co-meager. By our assumption $\dim_{tH} B_n = \dim_{tH} K > \dim_{tH} K - \varepsilon$ (which also implies $\dim_t B_n > 0$ by Fact 2.1, since $\dim_{tH} K \geq 1$ and

$\varepsilon < 1$), thus Theorem 2.71 yields that $\mathcal{B}_n$ is co-meager in $C(B_n)$. Lemma 2.75 implies that $R_n^{-1}(\mathcal{B}_n)$ is co-meager in $C(K)$ for all $n \in \mathbb{N}$, thus $\mathcal{F}_\varepsilon$ is also co-meager.

It remains to show that for every $f \in \mathcal{F}_\varepsilon$ the set $L_{f,\varepsilon}$ is co-meager in $f(K)$. Let us fix $f \in \mathcal{F}_\varepsilon$. We will actually show that $L_{f,\varepsilon}$ contains an open set in $\mathbb{R}$ which is a dense subset of $f(K)$. So let $U \subseteq \mathbb{R}$ be an open set in $\mathbb{R}$ such that $f(K) \cap U \neq \emptyset$. It is enough to prove that $L_{f,\varepsilon} \cap U$ contains an interval. Since the B_n's form a basis, the continuity of f implies that there exists an $n \in \mathbb{N}$ such that $f(B_n) \subseteq U$. It is easy to see using the definition of $\mathcal{F}_\varepsilon$ that $f|_{B_n} \in \mathcal{B}_n$, so there exists a non-degenerate interval $I_{f|_{B_n},\varepsilon}$ such that for all $y \in I_{f|_{B_n},\varepsilon}$ we have

$$\dim_H f^{-1}(y) \geq \dim_H \left(f|_{B_n}\right)^{-1}(y) \geq \dim_{tH} K - 1 - \varepsilon.$$

Thus $I_{f|_{B_n},\varepsilon} \subseteq L_{f,\varepsilon}$. On the other hand, as we saw above, $\dim_{tH} K - \varepsilon > 0$. Hence, $\dim_{tH} K - 1 - \varepsilon > -1$ which implies $\left(f|_{B_n}\right)^{-1}(y) \neq \emptyset$ for every $y \in I_{f|_{B_n},\varepsilon}$, thus $I_{f|_{B_n},\varepsilon} \subseteq f(B_n)$. But it follows from $f(B_n) \subseteq U$ that $I_{f|_{B_n},\varepsilon} \subseteq U$. Hence $I_{f|_{B_n},\varepsilon} \subseteq L_{f,\varepsilon} \cap U$ and this completes the proof. $\square$

2.6 Open Problems

First let us recall the most interesting open problem.

Problem 2.32. *Determine the almost sure topological Hausdorff dimension of the trail of the d-dimensional Brownian motion for $d = 2$ or 3.*

Now we collect a few more open problems.

Problem 2.76. *Let $B \subseteq \mathbb{R}^d$ be a Borel set and $\varepsilon > 0$. Does there exist a compact set $K \subseteq B$ with $\dim_{tH} K \geq \dim_{tH} B - \varepsilon$?*

Problem 2.77. *Let $B \subseteq \mathbb{R}^d$ be a Borel set and $1 \leq c < \dim_{tH} B$ arbitrary. Does there exist a Borel set $B' \subseteq B$ with $\dim_{tH} B' = c$?*

The next problem is somewhat vague. It is motivated by the proof of Theorem 2.34.

Problem 2.78. *Is there some sort of structural characterization of the sets with topological Hausdorff dimension at least c? For example, is it true that a Borel set $B \subseteq \mathbb{R}^d$ satisfies $\dim_{tH} B \geq c$ iff it contains a disjoint family of non-degenerate connected sets such that each set meeting all members of this family is of Hausdorff dimension at least $c - 1$?*

In a somewhat similar vein, is there some sort of analogue of Frostman's Lemma? (See e.g. [14] or [29].)

Moreover, it would also be interesting to know whether the theory of the topological packing dimension and topological box-counting dimension (defined analogously to $\dim_{tH} B$ in the obvious way) differs significantly from ours.

Chapter 3

Duality in LCA Polish groups

3.1 Outline

As we stated in the Introduction, the goal of the chapter is to prove the following theorem.

Theorem 3.1. *There is no addition preserving Erdős–Sierpiński mapping on any uncountable LCA Polish group.*

Let G be a locally compact abelian (LCA) Polish group. Let $\mathcal{M}$ and $\mathcal{N}$ be the ideals of meager and null (with respect to Haar measure) subsets of G. Let $(\varphi_{\mathcal{M}})$ denote the following statement (considered by Carlson in [9]): For every $S \in \mathcal{M}$ there exists a set $S' \in \mathcal{M}$ such that

$$\forall x_1, x_2 \in G \, \exists x \in G \quad (S + x_1) \cup (S + x_2) \subseteq S' + x.$$

Let $(\varphi_{\mathcal{N}})$ be the dual statement obtained by replacing $\mathcal{M}$ by $\mathcal{N}$.

If there exists an Erdős–Sierpiński mapping preserving addition then $(\varphi_{\mathcal{M}})$ and $(\varphi_{\mathcal{N}})$ are equivalent. In Section 3.2 we show that $(\varphi_{\mathcal{M}})$ holds in LCA Polish groups. In Section 3.3 we begin to show that $(\varphi_{\mathcal{N}})$ fails for all uncountable LCA Polish groups by reducing the general case to three special cases. Finally, in Section 3.4 we settle these three special cases.

3.2 $(\varphi_{\mathcal{M}})$ holds for all LCA Polish groups

As the known proofs only work for the reals, we had to come up with a new, topological proof.

Notation 3.2. Let X be a metric space, $x \in X$ and $r > 0$. Let $B(x,r)$ denote the closed ball of radius r centered at the point x.

Lemma 3.3. *Let X be a metric space and $C \subseteq X$ a nowhere dense compact set. Then there exists a function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x \in X$ and $r > 0$ there exists $y \in X$ such that $B(y, f(r)) \subseteq B(x,r) \setminus C$.*

Proof. Suppose towards a contradiction that $r > 0$ is such a number that there exists a sequence $r_n \to 0$ $(n \in \mathbb{N})$ and a set $\{x_n \in X : n \in \mathbb{N}\}$ such that for all $y \in X$ and $n \in \mathbb{N}$

$$B(y, r_n) \nsubseteq B(x_n, r) \setminus C. \tag{3.1}$$

$r_n < r$ holds for large enough n, so in the case $y = x_n$ we obtain that for large enough n there exist $z_n \in B(x_n, r_n) \cap C$. By the compactness of C there exists a convergent subsequence $\lim_{k\to\infty} z_{n_k} = z \in C$, and so $\lim_{k\to\infty} x_{n_k} = z \in C$. There is an $N \in \mathbb{N}$ such that $x_{n_k} \in B(z, \frac{r}{2})$ holds for all $k > N$. Then $B\left(z, \frac{r}{2}\right) \subseteq B(x_{n_k}, r)$, so by (3.1) for all $k > N$ and $y \in B\left(z, \frac{r}{2}\right)$ we obtain $B(y, r_{n_k}) \nsubseteq B\left(z, \frac{r}{2}\right) \setminus C$, which contradicts that C is nowhere dense, and we are done. $\square$

Every abelian Polish group admits a compatible invariant complete metric, because it admits a compatible invariant metric by [21, Thm. 7.3.], and a compatible invariant metric is automatically complete by [21, Lem. 7.4.]. So we may assume that the metric on our group is invariant.

Lemma 3.4. *Let G be an abelian Polish group and $C \subseteq G$ a nowhere dense compact set. Then there exists a function $l\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, x_1, x_2 \in G$ and $r > 0$ there is a $y \in G$ such that*

$$B(y, l(r)) \subseteq B(x,r) \setminus ((C + x_1) \cup (C + x_2)).$$

Proof. It is easy to see, using the invariance of our metric, that $l = f \circ f$ works, where f is defined in the previous lemma. $\square$

We may assume that $l(r) < r$ for all $r > 0$. Now we prove $(\varphi_{\mathcal{M}})$.

Theorem 3.5. *Let S be a meager set in an LCA Polish group G. Then there is a meager set $T \subseteq G$ such that for all $s_1, s_2 \in G$ there is a $t \in G$ such that $(S+s_1) \cup (S+s_2) \subseteq T+t$.*

Proof. We may assume by local compactness (by taking closures of the nowhere dense subsets and decomposing each of them to countably many compact sets) that $S = \cup_{n\in\mathbb{N}} S_n$, where the S_n's $(n \in \mathbb{N})$ are nowhere dense compact sets. The idea is that we construct nowhere dense T_n's for the S_n's simultaneously, and T will be the union of the T_n's. Later we will set $T_n = \cap_{k\in\mathbb{N}} T_n^k$ for some open sets T_n^k, and we will simultaneously construct a decreasing sequence of closed balls $B(x_k, r_k)$ such that $t = t_{s_1,s_2}$ will be found as $\cap_{k\in\mathbb{N}} B(x_k, r_k)$.

Let $h\colon \mathbb{N} \to \mathbb{N}\times\mathbb{N}$ be an enumeration of $\mathbb{N}\times\mathbb{N}$, let $h_1, h_2\colon \mathbb{N}\to\mathbb{N}$ be the first and second coordinate functions of h and let $\{g_n : n\in\mathbb{N}\}$ be a dense set in G. Let l_n be the function in the previous lemma for the compact set S_n. We define T by induction, independently from s_1, s_2. Let $x_0 \in G$ be arbitrary and $r_0 > 0$ so small that $B(x_0, r_0)$ is compact (we can do this by local compactness of G), and $r_k = \frac{l_{h_1(k)}(r_{k-1})}{2}$ if $k > 0$ and $n \in \mathbb{N}$. Let

$$T_n^k = \begin{cases} G & \text{if } n \neq h_1(k), \\ G \setminus B(g_{h_2(k)}, r_k) & \text{if } n = h_1(k). \end{cases} \tag{3.2}$$

The key step of the proof is the following lemma.

Lemma 3.6. *Assume that $x_0, x_1, \ldots, x_{k-1}$ are already defined. Then there exists $x_k \in G$ (which depends on s_1 and s_2) such that $B(x_k, r_k) \subseteq B(x_{k-1}, r_{k-1})$ and for all $n \in \mathbb{N}$ and $t \in B(x_k, r_k)$*

$$(S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t. \tag{3.3}$$

Proof of the Lemma. If $n = h_1(k)$, then $T_n^k = G \setminus B(g_{h_2(k)}, r_k)$. Applying the equation $2r_k = l_{h_1(k)}(r_{k-1}) = l_n(r_{k-1})$ and Lemma 3.4 there exists $y_k \in B(x_{k-1} + g_{h_2(k)}, r_{k-1})$ such that

$$B(y_k, 2r_k) \subseteq B(x_{k-1} + g_{h_2(k)}, r_{k-1}) \setminus ((S_n + s_1) \cup (S_n + s_2)). \tag{3.4}$$

Let us define $x_k = y_k - g_{h_2(k)}$. Using the definition of x_k and (3.4) we obtain that

$$B(x_k, r_k) = B(y_k, r_k) - g_{h_2(k)} \subseteq B(y_k, 2r_k) - g_{h_2(k)} \subseteq B(x_{k-1}, r_{k-1}).$$

We will use the following easy equation, where the first $+$ is the Minkowski sum.

$$B(g_{h_2(k)}, r_k) + B(x_k, r_k) \subseteq B(g_{h_2(k)} + x_k, 2r_k). \tag{3.5}$$

Using (3.4) again, the definition of x_k, and (3.5) in this order we obtain that for all $t \in B(x_k, r_k)$

$$\begin{aligned}(S_n + s_1) \cup (S_n + s_2) &\subseteq G \setminus B(y_k, 2r_k) = G \setminus B(g_{h_2(k)} + x_k, 2r_k) \\ &\subseteq G \setminus B(g_{h_2(k)}, r_k) + t.\end{aligned}$$

Hence

$$(S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t,$$

so (3.3) holds for $n = h_1(k)$. If $n \neq h_1(k)$ then $T_n^k = G$ and (3.3) is obvious, so we are done. □

Now we return to the proof of Theorem 3.5. By the compactness of $B(x_0, r_0)$ the closed sets $B(x_k, r_k)$ are compact, so the intersection of decreasing sequence of compact sets $\cap_{k\in\mathbb{N}} B(x_k, r_k) \neq \emptyset$. Let

$$t_{s_1,s_2} \in \cap_{k\in\mathbb{N}} B(x_k, r_k)$$

be the common shift. By (3.3) and the definition of t_{s_1,s_2},

$$(S_n + s_1) \cup (S_n + s_2) \subseteq T_n^k + t_{s_1,s_2} \tag{3.6}$$

holds for all $k, n \in \mathbb{N}$. For every $n \in \mathbb{N}$ the set $T_n = \cap_{k\in\mathbb{N}} T_n^k$ is nowhere dense by (3.2). By (3.6) we easily obtain for every $n \in \mathbb{N}$

$$(S_n + s_1) \cup (S_n + s_2) \subseteq T_n + t_{s_1,s_2}.$$

The set $T = \cup_{n\in\mathbb{N}} T_n$ is meager, and clearly

$$(S + s_1) \cup (S + s_2) \subseteq T + t_{s_1,s_2},$$

and the proof is complete. □

3.3 Reduction to $\mathbb{T}$, $\mathbb{Z}_p$, and $\prod_{n\in\mathbb{N}} G_n$

In this section we reduce the general uncountable LCA Polish groups to some special groups. We follow the strategy developed in [11].

Definition 3.7. Let us say that an LCA Polish group G is *nice* if $(\varphi_{\mathcal{N}})$ fails in G, that is the following. There is a nullset N such that for every nullset N' there are $g_1, g_2 \in G$ such that

$$\forall g \in G \quad (N + g_1) \cup (N + g_2) \nsubseteq (N' + g).$$

Lemma 3.8. *If an LCA Polish group G has a nice open subgroup U then G is nice.*

Proof. Let ν be the Haar measure of the open subgroup $U \subseteq G$ and $N_U \subseteq U$ be a ν-null set that witnesses that U is nice. As U is an open subgroup and G is separable, we can write a disjoint countable decomposition $G = \cup_{n=0}^{\infty} (U + g_n)$. It is easy to see that $\mu(B) = \sum_{n=0}^{\infty} \nu\left((B - g_n) \cap U\right)$ is a Haar measure on G. We show that the μ-null set $N = \cup_{n=0}^{\infty}(N_U + g_n)$ witnesses that G is nice. Let $N' \subseteq G$ be an arbitrary μ-null set. Clearly $N' \cap U \subseteq U$ is a ν-null set. Since N_U witnesses that U is nice, there are $u_1, u_2 \in U$ such that

$$\forall u \in U \quad (N_U + u_1) \cup (N_U + u_2) \nsubseteq (N' \cap U) + u. \tag{3.7}$$

It is enough to prove that

$$\forall g \in G \quad (N + u_1) \cup (N + u_2) \nsubseteq N' + g.$$

Suppose towards a contradiction that there is a $g \in G$ such that

$$(N + u_1) \cup (N + u_2) \subseteq N' + g. \tag{3.8}$$

There exists an $n \in \mathbb{N}$ such that $g - g_n \in U$. The definition of N and (3.8) imply

$$(N_U + u_1) \cup (N_U + u_2) \subseteq (N - g_n + u_1) \cup (N - g_n + u_2) \subseteq N' + (g - g_n). \tag{3.9}$$

Equation (3.9), $(N_U + u_1) \cup (N_U + u_2) \subseteq U$, and $g - g_n \in U$ yield

$$(N_U + u_1) \cup (N_U + u_2) \subseteq (N' \cap U) + (g - g_n).$$

Since $g - g_n \in U$, this contradicts (3.7). The proof is complete. □

Lemma 3.9. *Assume that G is an LCA Polish group, $C \subseteq G$ is a compact subgroup, and G/C is nice. Then G is also nice.*

Proof. If μ is a Haar measure on G and $\pi \colon G \to G/C$ is the canonical homomorphism then $\nu = \mu \circ \pi^{-1}$ is a Haar measure on G/C by [18, 63. Thm. C.]. Let $N \subseteq G/C$ be a ν-null set witnessing that G/C is nice. We prove that $\pi^{-1}(N)$ witnesses that G is nice. Clearly $\mu\left(\pi^{-1}(N)\right) = \nu(N) = 0$, so $\pi^{-1}(N) \subseteq G$ is a μ-null set. Assume to the contrary that there exists a μ-null set $N' \subseteq G$ such that for all $g_1, g_2 \in G$ there exists $g \in G$ such that

$$\left(\pi^{-1}(N) + g_1\right) \cup \left(\pi^{-1}(N) + g_2\right) \subseteq N' + g. \tag{3.10}$$

The left hand side of (3.10) are composed of cosets of C, so we may assume by shrinking N' that N' consists of cosets, too. Then clearly $\pi(N') \subseteq G/C$ is a ν-null set. Since π is a homomorphism, (3.10) infers

$$(N + \pi(g_1)) \cup (N + \pi(g_2)) \subseteq \pi(N') + \pi(g). \tag{3.11}$$

If (g_1, g_2) ranges over G^2 then $(\pi(g_1), \pi(g_2))$ ranges over $(G/C)^2$. Thus (3.11) contradicts that N is a witness that G/C is nice. □

Now we start reducing the problem to simpler groups.

Definition 3.10. Let G be a group and p be a prime number, $G_{p^n} = \{g \in G : p^n g = 0\}$ for every $n \in \mathbb{N}$, and also let $G_{p^\infty} = \cup_{n\in\mathbb{N}} G_{p^n}$. We say that G is a *p-group*, if $G = G_{p^\infty}$.

Definition 3.11. Let p be a prime number. An abelian group G is called *quasicyclic* if it is generated by a sequence $\langle g_n \rangle_{n\in\mathbb{N}}$ with the property that $g_0 \neq 0$ and $pg_{n+1} = g_n$ for every $n \in \mathbb{N}$. For a fixed prime p the unique (up to isomorphism) quasicyclic group is denoted by C_{p^∞}.

Notation 3.12. We denote by $\mathbb{T}$ the circle group, by $\mathbb{Z}_p$ the group of p-adic integers for every prime p, and by $\prod_{n\in\mathbb{N}} G_n$ the product of the finite abelian groups G_n.

Remark 3.13. $\mathbb{Z}_p$ is the topological space $\{0, 1, \ldots, p-1\}^\omega$ equipped with the product topology. Addition is coordinatewise with carried digits from the n^{th} coordinate to the $n+1^{st}$. Both $\mathbb{Z}_p$ and $\prod_{n\in\mathbb{N}} G_n$ are Polish with the product topology.

We will use the following theorem from [11].

Theorem 3.14. *Every infinite abelian group contains a subgroup isomorphic to one of the following:*

1. $\mathbb{Z}$,
2. C_{p^∞} *for some prime* p,
3. $\bigoplus_{n\in\mathbb{N}} G_n$, *where each* G_n *is a finite abelian group of at least two elements.*

Theorem 3.15. $\mathbb{T}$, $\mathbb{Z}_p$, *and* $\prod_{n\in\mathbb{N}} G_n$ *are nice.*

Proof. We postpone the proof to the next section. □

Theorem 3.16. *Every uncountable LCA Polish group is nice, that is,* (φ_N) *fails for every uncountable LCA Polish group.*

Proof. Let G be an LCA Polish group. From the Principal Structure Theorem of LCA groups [34, 2.4.1] follows that G has an open subgroup U which is of the form $U = K \otimes \mathbb{R}^n$, where K is a compact subgroup and $n \in \mathbb{N}$. G is uncountable, so nondiscrete, and U is open, therefore U is a nondiscrete, so uncountable group. By Lemma 3.8 it is enough to prove that U is nice, so we can assume $G = U$.

Assume that $n \geq 1$. $\mathbb{R}$ is nice by [26], let N be a nullset witnessing this fact. It is obvious that $K \times N \times \mathbb{R}^{n-1}$ witnesses that $G = K \otimes \mathbb{R}^n$ is nice. Therefore we may assume $n = 0$, and we obtain that G is compact. It is enough to find a closed subgroup $C \subseteq G$ such that G/C is nice by Lemma 3.9. Using [34, 2.1.2] and the Pontryagin Duality Theorem [34, 1.7.2], we obtain that factors of G are the same as (isomorphically homeomorphic to) dual groups of closed subgroups of $\widehat{G}$. If G is compact, then $\widehat{G}$ is discrete, see [34, 1.2.5]. Hence it is enough to find a subgroup $H \subseteq \widehat{G}$ such that $\widehat{H}$ is nice.

From the previous theorem follows (using that $G = K$ is an infinite group) that $\widehat{G}$ has a subgroup isomorphic either to $\mathbb{Z}$, or to C_{p^∞} for some prime p, or to $\oplus_{n\in\mathbb{N}} G_n$ (where each G_n is a finite abelian group of at least two elements). It is sufficient to show that the duals of these groups are nice. It is well-known that $\widehat{\mathbb{Z}} = \mathbb{T}$, and $\mathbb{T}$ is nice by Theorem 3.15.

By [34, 2.2.3] the dual of a direct sum (equipped with the discrete topology) is the direct product of the dual groups (equipped with the product topology), so $\widehat{\oplus_{n\in\mathbb{N}} G_n} = \prod_{n\in\mathbb{N}} \widehat{G_n}$. If G_n is finite so is $\widehat{G_n}$, thus we are done with the second case by Theorem 3.15.

Finally, $\widehat{C_{p^\infty}} = \mathbb{Z}_p$, see [20, 25.2]. Hence Theorem 3.15 completes the proof. □

3.4 $(\varphi_{\mathcal{N}})$ fails for $\mathbb{T}$, $\mathbb{Z}_p$, and $\prod_{n\in\mathbb{N}} G_n$

We will prove that $\mathbb{T}$, $\mathbb{Z}_p$, and $\prod_{n\in\mathbb{N}} G_n$ are nice. $\mathbb{T}$ is nice, see [26], so we need to handle the last two cases. Our proofs are very similar to the Main Theorem of Kysiak's paper, see [26, Main Thm. 4.3.]. The only important difference is the proof of Lemma 3.22, we need a new idea, Kysiak's arguments are not applicable here. He uses a specific relation between the metric and the addition in $\mathbb{T}$ that fails in $\mathbb{Z}_p$.

Theorem 3.17. *$\mathbb{Z}_p$ is nice.*

Proof. We use the description of $\mathbb{Z}_p$ that can be found in Remark 3.13. Let μ be the Haar measure on $\mathbb{Z}_p$, and let $\mu_2 = \mu \times \mu$ be the Haar measure on $\mathbb{Z}_p^2$. The following lemma is analogous to [26, Lemma 4.7.], its proof can be repeated verbatim, only write $\mathbb{Z}_p$ instead of $(0,1]$ and use the previous definitions of μ and μ_2.

Lemma 3.18. *Let J' be a Borel subset of $\mathbb{Z}_p$, and consider $\delta = 1 - \mu(J')$ and $\varepsilon > 0$. Then the set $S = \{t \in \mathbb{Z}_p : \mu(J' \cup (t + J')) > 1 - \varepsilon\}$ is Haar measurable, and*

$$\mu(S) \geq 1 - \frac{\delta^2}{\varepsilon}.$$

Now we return to the proof of Theorem 3.17. Fix a partition of the set ω into consecutive disjoint intervals I_n ($n \in \mathbb{N}^+$) such that the number of its elements is $|I_n| = 2n$. Then it is easy to see that

$$\frac{1}{p^{|I_n|}} < \frac{1}{n^2}. \tag{3.12}$$

Inequality (3.12) follows that we can choose sets $J_n \subseteq p^{I_n}$ for all $n \in \mathbb{N}^+$ such that

$$1 - \frac{4}{n^2} < \frac{|J_n|}{p^{|I_n|}} < 1 - \frac{3}{n^2}, \tag{3.13}$$

and J_n consist of the first $|J_n|$ consecutive elements of p^{I_n} with respect to the antilexicographical ordering (sequences are ordered according to the largeness of the rightmost coordinate where they differ).

Let $\mathcal{N}^*$ be the filter of full measure sets of $\mathbb{Z}_p$. If $I \subseteq \omega$ and $J \subseteq p^I$ then $[J]$ denote the set $\{x \in \mathbb{Z}_p : x \restriction I \in J\}$. For $s \in p^{<\omega}$ let $[s] = \{x \in \mathbb{Z}_p : x \text{ extends } s\}$. Let us define

$$N = \bigcap_{m=1}^{\infty} \bigcup_{n>m} (\mathbb{Z}_p \setminus [J_n]) \quad \text{and} \quad A = \mathbb{Z}_p \setminus N.$$

Equation (3.13) follows $\sum_{n=1}^{\infty} \mu(\mathbb{Z}_p \setminus [J_n]) \leq \sum_{n=1}^{\infty} \frac{4}{n^2} < \infty$, so the Borel–Cantelli Lemma implies $\mu(N) = 0$. We show that N witnesses that $\mathbb{Z}_p$ is nice. It is enough to prove that for every $B \in \mathcal{N}^*$ there are $x_1, x_2 \in \mathbb{Z}_p$ such that

$$\forall x \in \mathbb{Z}_p \quad (B + x_1) \cup (B + x_2) \nsubseteq A + x, \tag{3.14}$$

because it is easy to see that (3.14) is equivalent with

$$\forall x \in \mathbb{Z}_p \quad (N - x_1) \cup (N - x_2) \nsubseteq (\mathbb{Z}_p \setminus B) + x.$$

In order to prove (3.14) we fix $x_1 = 0$ and construct $y_0 \in \mathbb{Z}_p$ such that

$$\forall x \in \mathbb{Z}_p \quad B \cup (y_0 + B) \nsubseteq x + A.$$

For this purpose let us fix any closed set $C \subseteq B$ of positive measure. We find an $y_0 \in \mathbb{Z}_p$ such that

$$\forall x \in \mathbb{Z}_p \quad C \cup (y_0 + C) \nsubseteq x + A. \tag{3.15}$$

Without loss of generality we may assume that for every $s \in p^{<\omega}$ we have $[s] \cap C = \emptyset$ or $\mu([s] \cap C) > 0$ (if not, consider $C' = C \setminus \bigcup \{[s] : s \in p^{<\omega} \text{ and } \mu([s] \cap C) = 0\}$ instead).

Lemma 3.19. *Let $\lambda_n = 1 - \mu([J_n])$ and $\varepsilon_n = \frac{1}{4}\lambda_n$. We inductively define* *(i) A strictly increasing sequence of positive integers* $\langle n_k\rangle_{k\in\mathbb{N}^+}$,

(ii) Sets $J'_{n_k} \subseteq p^{I_{n_k}}$ *such that for every* $c \in C$ *and* $u \in J'_{n_{k+1}}$ *there is a* $c' \in C$ *such that* $c' \restriction (I_1 \cup \cdots \cup I_{n_k}) = c \restriction (I_1 \cup \cdots \cup I_{n_k})$ *and* $c' \restriction I_{n_{k+1}} = u$,

(iii) Sets $S_k \subseteq \mathbb{Z}_p$ $(k \geq 2)$ *such that* $\mu(S_k) > 1 - \frac{1}{4^k}$ *and*

$$\forall t \in S_k \quad \mu\left([J'_{n_k}] \cup \left(t + [J'_{n_k}]\right)\right) > 1 - \varepsilon_{n_k}.$$

Proof of Lemma 3.19. The proof is by induction on k. For $k = 1$ let $n_1 = 1$ and $J'_1 = C \restriction I_1 \subseteq \{0, \ldots, p-1\}^2$. Let us fix an arbitrary k, and assume that we have already defined n_k. It is enough to construct n_{k+1}, $J'_{n_{k+1}}$, and S_{k+1}. Let $C \restriction (I_1 \cup \cdots \cup I_{n_k}) = \{s_1, \ldots, s_l\}$ be a finite set of restrictions of elements of C. Let us recall that $[s_i] \cap C$ has positive measure for every $i \in \{1, \ldots, l\}$. We prove that there is an $m > n_k$ such that

$$P = p^{I_1 \cup \cdots \cup I_m} \times \bigcap_{i=1}^{l} \{x \restriction (\max I_m, \infty) : x \in C \text{ and } x \text{ extends } s_i\} \subseteq \mathbb{Z}_p$$

has positive measure. To see this, it is enough to prove that every $s \in \{s_1, \ldots, s_l\}$ has an extension $r_s \in p^{I_1 \cup \cdots \cup I_m}$ such that

$$\mu\left([r_s] \cap C\right) > \frac{l-1}{l}\mu\left([r_s]\right), \tag{3.16}$$

because then

$$p^{I_1 \cup \cdots \cup I_m} \times \bigcap_{i=1}^{l} \{x \restriction (\max I_m, \infty) : x \in C \text{ and } x \text{ extends } r_{s_i}\}$$

is a subset of P with positive measure. Equation (3.16) follows from the following density theorem applying for $X = G$ and $V_n = [0^n]$. The proof of the next theorem can be found in [16, 447D Thm.].

Theorem 3.20. *Assume that* X *is a topological group with a left Haar measure* μ, *and let* $\langle V_n\rangle_{n\in\mathbb{N}}$ *be a non-increasing sequence of closed neighborhoods of the identity, constituting a base of neighborhoods of the identity with the following property. There exists an* $M \in \mathbb{N}$ *such that for all* $n \in \mathbb{N}$ *the set* $V_n - V_n$ *can be covered by at most* M *translates of* V_n. *Then for every Haar measurable set* $E \subseteq X$ *for* μ *almost every* $x \in X$ *we have*

$$\lim_{n\to\infty} \frac{\mu\left(E \cap (x + V_n)\right)}{\mu(V_n)} = \chi_E(x).$$

Now we return to the proof of Lemma 3.19. For $n > m$ consider

$$J'_n = \{x \restriction I_n : x \in P\} \quad \text{and} \quad \delta_n = 1 - \mu\left([J'_n]\right).$$

We have

$$P \subseteq \{x \in \mathbb{Z}_p : \forall n > m \quad x \restriction I_n \in J'_n\},$$

so the set on the right hand side has positive measure. Thus we obtain $\prod_{n=m+1}^{\infty} \mu\left([J'_n]\right) = \prod_{n=m+1}^{\infty}(1-\delta_n) > 0$. Hence $\sum_{n=m+1}^{\infty} \delta_n < \infty$, so for infinitely many $n > m$ we have

$$\delta_n < \frac{1}{n2^{k+2}}. \tag{3.17}$$

Let n_{k+1} be the first $n > m$ that satisfies (3.17). Then (ii) follows from the definition of $J'_{n_{k+1}}$. Equation (3.13) yields $\lambda_n = 1 - \mu\left([J_n]\right) > \frac{3}{n^2}$ for all $n \in \mathbb{N}^+$, so

$$4\varepsilon_{n_{k+1}} = \lambda_{n_{k+1}} > \frac{1}{n_{k+1}^2}. \tag{3.18}$$

Equations (3.17) and (3.18) imply

$$\frac{\delta_{n_{k+1}}^2}{\varepsilon_{n_{k+1}}} < \frac{1}{4^{k+1}}.$$

Let us apply Lemma 3.18 for the set

$$S_{k+1} = \left\{t \in \mathbb{Z}_p : \mu\left([J'_{n_{k+1}}] \cup \left(t + [J'_{n_{k+1}}]\right)\right) > 1 - \varepsilon_{n_{k+1}}\right\},$$

this implies $\mu(S_{k+1}) > 1 - \frac{\delta_{n_{k+1}}^2}{\varepsilon_{n_{k+1}}} > 1 - \frac{1}{4^{k+1}}$, and (iii) follows. □

Corollary 3.21. *For every choice of $u_k \in J'_{n_k}$ $(k \in \mathbb{N}^+)$ there exists $c \in C$ such that*

$$c \restriction I_{n_k} = u_k, \quad k \in \mathbb{N}^+.$$

Proof of Corollary 3.21. First we inductively define a sequence $\langle c_k \rangle_{k \in \mathbb{N}^+}$. The definition of J'_1 follows that there is a $c_1 \in C$ such that $c_1 \restriction I_1 = u_1$. If $c_k \in C$ is already defined, then Lemma 3.19 (ii) implies that there is a $c_{k+1} \in C$ such that $c_{k+1} \restriction (I_1 \cup \dots \cup I_{n_k}) = c_k \restriction (I_1 \cup \dots \cup I_{n_k})$ and $c_{k+1} \restriction I_{n_{k+1}} = u_{k+1}$.

Now let $c = \lim_{k \to \infty} c_k$, the limit exists by the construction of the c_k's. As $c_k \in C$ for every $k \in \mathbb{N}^+$ and C is closed, we obtain $c \in C$. Since $c_i \restriction I_{n_k} = u_k$ for all $i \geq k$, we have $c \restriction I_{n_k} = u_k$. □

Now we return to the proof of Theorem 3.17. The inequalities $\mu(S_k) > 1 - \frac{1}{4^k}$ follow that $\mu\left(\cap_{k=2}^{\infty} S_k\right) > 0$. Let us fix arbitrary $y_0 \in \cap_{k=2}^{\infty} S_k$ and $x_0 \in \mathbb{Z}_p$. According to (3.15) it is enough to check that

$$C \cup (y_0 + C) \nsubseteq x_0 + A. \tag{3.19}$$

Lemma 3.19 (iii) implies that $\mu\left([J'_{n_k}] \cup \left(y_0 + [J'_{n_k}]\right)\right) \geq 1 - \varepsilon_{n_k}$ for all $k \geq 2$, and clearly $\mu\left(x_0 + [J_{n_k}]\right) = 1 - \lambda_{n_k}$. Therefore for all $k \geq 2$ we have

$$\begin{aligned}\mu\left(\left([J'_{n_k}] \cup (y_0 + [J'_{n_k}])\right) \setminus (x_0 + [J_{n_k}])\right) &\geq (1 - \varepsilon_{n_k}) - (1 - \lambda_{n_k}) \\ &= \lambda_{n_k} - \varepsilon_{n_k}.\end{aligned}$$

Hence for every $k \geq 2$ either

$$\mu\left([J'_{n_k}] \setminus (x_0 + [J_{n_k}])\right) \geq \frac{\lambda_{n_k} - \varepsilon_{n_k}}{2}$$

or

$$\mu\left((y_0 + [J'_{n_k}]) \setminus (x_0 + [J_{n_k}])\right) \geq \frac{\lambda_{n_k} - \varepsilon_{n_k}}{2}.$$

As our proof will not refer to any specific properties of y_0, and the first inequality is a special case of the second one with $y_0 = 0$, we may assume that the second case holds for infinitely many k's. Let us denote the set of these k's by $\mathcal{K}$.

Lemma 3.22. *For every $k \in \mathcal{K}$ there is a $v_k \in J'_{n_k}$ such that for all $z \in \mathbb{Z}_p$*

$$z \restriction I_{n_k} = v_k \Rightarrow z \notin (x_0 - y_0) + [J_{n_k}].$$

Proof of Lemma 3.22. Assume that $k \in \mathcal{K}$ is fixed and let us define

$$L_k = \{((x_0 - y_0) + [J_{n_k}]) \restriction I_{n_k}\} \subseteq p^{I_{n_k}}.$$

The definition of L_k follows $(x_0 - y_0) + [J_{n_k}] \subseteq [L_k]$. Since the elements of J_{n_k} are consecutive (with respect to the antilexicographical ordering), we obtain $|L_k| \leq |J_{n_k}| + 1$. Hence the translation invariance of μ implies

$$\begin{aligned}\mu\left([L_k] \setminus ((x_0 - y_0) + [J_{n_k}])\right) &= \mu\left([L_k]\right) - \mu\left((x_0 - y_0) + [J_{n_k}]\right) \\ &= \mu\left([L_k]\right) - \mu\left([J_{n_k}]\right) \leq \frac{1}{p^{|I_{n_k}|}}.\end{aligned} \tag{3.20}$$

Let us recall that $\varepsilon_n = \frac{1}{4}\lambda_n$ and $\lambda_n > \frac{3}{n^2}$ by (3.13). Therefore the definition of $\mathcal{K}$ and (3.12) imply

$$\begin{aligned}\mu\left(\left(y_0 + [J'_{n_k}]\right) \setminus (x_0 + [J_{n_k}])\right) &\geq \frac{\lambda_{n_k} - \varepsilon_{n_k}}{2} > \frac{\lambda_{n_k}}{3} \\ &> \frac{1}{n_k^2} > \frac{1}{p^{|I_{n_k}|}}.\end{aligned}$$

The above inequality and the translation invariance of μ follow

$$\mu\left([J'_{n_k}] \setminus ((x_0 - y_0) + [J_{n_k}])\right) > \frac{1}{p^{|I_{n_k}|}}. \tag{3.21}$$

Finally, equations (3.20) and (3.21) yield $\mu\left([J'_{n_k}] \setminus [L_k]\right) > 0$. Thus $[J'_{n_k}] \setminus [L_k] \neq \emptyset$, so we can choose $v_k \in J'_{n_k} \setminus L_k$. The definition of L_k implies

$$z \restriction I_{n_k} = v_k \Rightarrow z \notin (x_0 - y_0) + [J_{n_k}],$$

so Lemma 3.22 follows. □

Now we return to the proof of Theorem 3.17. Let us choose for every $k \in \mathcal{K}$ a $v_k \in J'_{n_k}$ according to Lemma 3.22. By Corollary 3.21 we can fix $c_0 \in C$ such that $c_0 \restriction I_{n_k} = v_k$ for all $k \in \mathcal{K}$. Lemma 3.22 implies that $c_0 \notin (x_0 - y_0) + [J_{n_k}]$, that is $y_0 + c_0 \notin x_0 + [J_{n_k}]$ for all $k \in \mathcal{K}$. Thus the definition of N yields $y_0 + c_0 \in x_0 + N$. Clearly, $y_0 + c_0 \in (y_0 + C)$, so

$$y_0 + c_0 \in (y_0 + C) \cap (x_0 + N) = (y_0 + C) \setminus (x_0 + A).$$

Therefore (3.19) follows, and the proof of Theorem 3.17 is complete. □

The proof of the following theorem is very similar to the above one, so we only describe the necessary modifications.

Theorem 3.23. *$\prod_{n\in\mathbb{N}} G_n$ is nice.*

Proof. We write $\mathbb{N}$ and $\prod_{n\in\mathbb{N}} G_n$ instead of ω and $\mathbb{Z}_p$, respectively. Let μ be the Haar measure on $\prod_{n\in\mathbb{N}} G_n$, and let $\mu_2 = \mu \times \mu$ be the Haar measure on $\left(\prod_{n\in\mathbb{N}} G_n\right)^2$. The analogue of Lemma 3.18 has the same proof.

For the construction of I_n's we fix a partition of the set $\mathbb{N}$ instead of ω. Write $\prod_{i\in I_n} G_i$ and $|\prod_{i\in I_n} G_i|$ instead of p^{I_n} and $p^{|I_n|}$, respectively. At the choice of the sets J_n we omit the ordering condition.

Let $\mathcal{N}^*$ be the filter of full measure sets of $\prod_{n\in\mathbb{N}} G_n$. If $I \subseteq \mathbb{N}$ and $J \subseteq \prod_{n\in I} G_n$ then $[J]$ denote the set $\left\{x \in \prod_{n\in\mathbb{N}} G_n : x \restriction I \in J\right\}$. We write $\left\{\prod_{i\in I} G_i : |I| < \infty\right\}$ instead of $p^{<\omega}$ and for $s \in \left\{\prod_{i\in I} G_i : |I| < \infty\right\}$ let $[s] = \left\{x \in \prod_{n\in\mathbb{N}} G_n : x \text{ extends } s\right\}$.

For the definition of C we may assume that for every $s \in \left\{\prod_{i\in I} G_i : |I| < \infty\right\}$ we have $[s] \cap C = \emptyset$ or $\mu([s] \cap C) > 0$.

The analogue of Lemma 3.19 and its proof need the following modifications. Consider $\prod_{i\in I_1\cup\dots\cup I_m} G_i$ and $\prod_{i=\max I_m+1}^{\infty} G_i$ instead of $p^{I_1\cup\dots\cup I_m}$ and $p^{(\max I_m,\infty)}$, respectively.

The analogue of Lemma 3.22 has an easier proof than in the case $\mathbb{Z}_p$.

Lemma 3.24. *For every $k \in \mathcal{K}$ there is a $v_k \in J'_{n_k}$ such that for all $z \in \prod_{n\in\mathbb{N}} G_n$*

$$z \restriction I_{n_k} = v_k \Rightarrow z \notin (x_0 - y_0) + [J_{n_k}].$$

Proof of Lemma 3.24. Assume that $k \in \mathcal{K}$ is fixed and let us define

$$L_k = \{((x_0 - y_0) + [J_{n_k}]) \restriction I_{n_k}\} \subseteq \prod_{i \in I_{n_k}} G_i,$$

then obviously $(x_0 - y_0) + [J_{n_k}] = [L_k]$. Thus we have

$$\begin{aligned} \mu\left([J'_{n_k}] \setminus [L_k]\right) &= \mu\left([J'_{n_k}] \setminus ((x_0 - y_0) + [J_{n_k}])\right) \\ &\geq \frac{\lambda_{n_k} - \varepsilon_{n_k}}{2} > 0. \end{aligned}$$

Hence $J'_{n_k} \setminus L_k \neq \emptyset$, so we can choose $v_k \in J'_{n_k} \setminus L_k$. Then the definition of L_k implies

$$z \restriction I_{n_k} = v_k \Rightarrow z \notin (x_0 - y_0) + [J_{n_k}].$$

The proof of Lemma 3.24 is complete. □

Now we return to the proof of Theorem 3.23. The last few lines of the proof is the same as in Theorem 3.17. □

Remark 3.25. Theorem 3.23 could be proved easier based on Bartoszyński's paper [7], but we do not know to generalize Bartoszyński's method to the case $\mathbb{Z}_p$ in Theorem 3.17.

Chapter 4

The structure of rigid functions

4.1 The structure of rigid functions of one variable

4.1.1 Outline

Let us recall the following definitions.

Definition 4.1. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is called *vertically rigid*, if $\operatorname{graph}(cf)$ is isometric to $\operatorname{graph}(f)$ for all $c \in (0, \infty)$. (Clearly, $c \in \mathbb{R} \setminus \{0\}$ would be the same.)

We need the following technical generalizations.

Definition 4.2. If C is a subset of $(0, \infty)$ and $\mathcal{G}$ is a set of isometries of $\mathbb{R}^{d+1}$ then we say that f is vertically rigid *for a set* $C \subseteq (0, \infty)$ *via elements of* $\mathcal{G}$ if for every $c \in C$ there exists a $\varphi \in \mathcal{G}$ such that $\varphi(\operatorname{graph}(f)) = \operatorname{graph}(cf)$. (If we do not mention C or $\mathcal{G}$ then C is $(0, \infty)$ and $\mathcal{G}$ is the set of all isometries.)

In this section we study the vertically rigid functions of one variable, our starting point is Janković's conjecture.

Conjecture 4.3 (D. Janković). *A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.*

Later we consider Borel, Lebesgue, and Baire measurable functions, for more information see Section 1.4.

4.1.2 Proof of Janković's conjecture

Theorem 4.4 (Proof of Janković's conjecture). *A continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.*

Remark 4.5. In fact, our proof will show that it is sufficient if f is a continuous function that is vertically rigid for some uncountable set C.

It is of course very easy to see that these functions are vertically rigid and continuous. The proof of the difficult direction goes through three theorems, which are interesting in their own right. First we reduce the general case to translations, then the case of translations to horizontal translations, and finally we describe the continuous functions that are vertically rigid via horizontal translations.

Theorem 4.6. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous function vertically rigid for an uncountable set $C \subseteq (0,\infty)$. Then f is of the form $a + bx$ for some $a, b \in \mathbb{R}$ or f is vertically rigid for an uncountable set $D \subseteq (0,\infty)$* via translations.

Proof. Let φ_c be the isometry belonging to $c \in C$. First we show that we may assume that these isometries are orientation preserving. If uncountably many of the φ_c's are orientation preserving then we are done by shrinking C. Otherwise let $C' \subseteq C$ be uncountable so that $\varphi_{c'}$ is orientation reversing for every $c' \in C'$. Fix $c'_0 \in C'$, then one can easily check that $c'_0 f$ is vertically rigid via orientation preserving isometries for $C'' = \left\{\frac{c'}{c'_0} : c' \in C'\right\}$. Suppose that we have already proved the theorem in case all isometries are orientation preserving. Then either $c'_0 f$ is of the form $a + bx$, and then so is f, or $c'_0 f$ is vertically rigid for an uncountable set D *via translations*, but then so is f itself (for the same set D, but possibly different translations).

For a function f let S_f be the set of directions between pairs of points on the graph of f, that is,

$$S_f := \left\{ \frac{p-q}{|p-q|} : p, q \in \operatorname{graph}(f),\ p \neq q \right\}.$$

Clearly S_f is a symmetric (about the origin) subset of the unit circle $\mathbb{S}^1 \subseteq \mathbb{R}^2$. As f is a function, $(0, \pm 1) \notin S_f$. Since f is continuous, it is easy to see that S_f actually consists of two (possibly degenerate) nonempty intervals. (Indeed, if $p = (x, f(x))$ and $q = (y, f(y))$ then $x < y$ and $x > y$ define two connected sets, open half planes in $\mathbb{R}^2$, whose continuous images form S_f.)

An orientation preserving isometry φ of the plane is either a translation or a rotation. Denote by $\operatorname{ang}(\varphi)$ the angle of φ in case it is a rotation, and set $\operatorname{ang}(\varphi) = 0$ if φ is a translation.

Now we define two self-maps of $\mathbb{S}^1$. Denote by ϱ_α the rotation about the origin by angle α. For $c > 0$ let ψ_c be the map obtained by 'multiplying by c', that is, let

$$\psi_c((x,y)) = \frac{(x, cy)}{|(x, cy)|} \quad ((x,y) \in \mathbb{S}^1).$$

It is easy to see that the rigidity of f implies that for every $c \in C$

$$S_f = \varrho_{\mathrm{ang}(\varphi_c)}(\psi_c(S_f)). \tag{4.1}$$

If S_f consists of two points, then f is clearly of the form $a + bx$ and we are done.

Let now $S_f = I \cup -I$, where I is a subinterval of $\mathbb{S}^1$ in the right half plane. We claim that the endpoints of I are among $(0, \pm 1)$ and $(1, 0)$. Suppose this fails, and consider the function $l(c) = \mathrm{arclength}\,(\psi_c(I))$ $(c \in (0, \infty))$. It is easy to see that l is real analytic, and we show that it is not constant. Let us first assume that $(0, 1)$ and $(0, -1)$ are not endpoints of I, then $\lim_{c \to 0} l(c) = 0$, so l cannot be constant (as $l > 0$). Let us now suppose that either $(0, 1)$ or $(0, -1)$ is an endpoint of I, then $0 < \mathrm{arclength}(I) < \frac{\pi}{2}$ or $\frac{\pi}{2} < \mathrm{arclength}(I) < \pi$. In both cases $\lim_{c \to 0} l(c) = \frac{\pi}{2}$ but $l(c) \neq \frac{\pi}{2}$, so l is not constant. As l is analytic, it attains each of its values at most countably many times, so there exists a $c \in C$ so that $\mathrm{arclength}\,(\psi_c(I)) \neq \mathrm{arclength}(I)$, which contradicts (4.1).

(Actually, it can be shown by a somewhat lengthy calculation using the derivatives that l attains each value at most twice.)

But this easily yields $\mathrm{ang}(\varphi_c) = 0$ or π for every $c \in C$. (Note that $(0, \pm 1) \notin S_f$ and that S_f is symmetric.) Just as above, we may assume that $\mathrm{ang}(\varphi_c) = 0$ for all $c \in C$. (Indeed, choose C' and c_0' analogously.) But then f is vertically rigid for an uncountable set *via translations*, so the proof is complete. □

The following theorem will be also useful in the two-dimensional case, so we formalize it for $\mathbb{R}^d$ instead of $\mathbb{R}$. We use the notation $\vec{x}$ for vectors of $\mathbb{R}^d$.

Theorem 4.7. *Let $f \colon \mathbb{R}^d \to \mathbb{R}$ be an arbitrary function that is vertically rigid for a set $C \subseteq (0, \infty)$* via translations. *Then there exists $a \in \mathbb{R}$ such that $f - a$ is vertically rigid for the same set via horizontal translations.*

Proof. We can clearly assume that $1 \notin C$. By assumption, for every $c \in C$ there exists $\vec{u}_c, \vec{v}_c \in \mathbb{R}^d$ such that

$$cf(\vec{x}) = f\left(\vec{x} + \vec{u}_c\right) + \vec{v}_c \;\; \left(\forall \vec{x} \in \mathbb{R}^d\right) \tag{4.2}$$

Applying this first with $c = c_2$, then with $c = c_1$ we obtain

$$\begin{aligned} c_1 c_2 f\left(\vec{x}\right) &= c_1 \left(f\left(\vec{x} + \vec{u}_{c_2}\right) + \vec{v}_{c_2}\right) = c_1 f\left(\vec{x} + \vec{u}_{c_2}\right) + c_1 \vec{v}_{c_2} \\ &= f\left(\vec{x} + \vec{u}_{c_1} + \vec{u}_{c_2}\right) + \vec{v}_{c_1} + c_1 \vec{v}_{c_2}. \end{aligned} \tag{4.3}$$

Interchanging c_1 and c_2 we obtain

$$c_2 c_1 f(\vec{x}) = f\left(\vec{x} + \vec{u}_{c_2} + \vec{u}_{c_1}\right) + \vec{v}_{c_2} + c_2 \vec{v}_{c_1}. \tag{4.4}$$

Comparing (4.3) and (4.4) yields $\vec{v}_{c_1} + c_1\vec{v}_{c_2} = \vec{v}_{c_2} + c_2\vec{v}_{c_1}$, so

$$\frac{\vec{v}_{c_1}}{c_1 - 1} = \frac{\vec{v}_{c_2}}{c_2 - 1} \quad \text{for all } c_1, c_2 \in C,$$

consequently $a = \frac{\vec{v}_c}{c-1}$ is the same value for all $c \in C$. Substituting this back to (4.2) gives $cf(\vec{x}) = f(\vec{x} + \vec{u}_c) + (c-1)a$, so $c(f(\vec{x}) - a) = f(\vec{x} + \vec{u}_c) - a$ for all $c \in C$, hence $f - a$ is vertically rigid for C via horizontal translations. □

Definition 4.8. For a function $f\colon \mathbb{R}^d \to \mathbb{R}$ and a set $C \subseteq (0, \infty)$ let $T_{f,C} \subseteq \mathbb{R}^d$ be the additive group generated by the set $T' = \left\{\vec{t} \in \mathbb{R}^d : \exists c \in C\ \forall \vec{x} \in \mathbb{R}^d\ f\left(\vec{x} + \vec{t}\right) = cf(\vec{x})\right\}$. (We will usually simply write T for $T_{f,C}$.)

Lemma 4.9. *Let $f\colon \mathbb{R}^d \to \mathbb{R}$ be a vertically rigid function for a set $C \subseteq (0, \infty)$ via horizontal translations such that $f(\vec{0}) = 1$. Then*

$$f\left(\vec{x} + \vec{t}\right) = f(\vec{x})\, f\left(\vec{t}\right)\ \ \forall \vec{x} \in \mathbb{R}^d,\ \forall \vec{t} \in T.$$

Moreover, $f\left(\vec{t}\right) > 0$ for every $\vec{t} \in T$, and T' is uncountable if so is C.

Proof. By assumption, for every $c \in C$ there exists $\vec{t}_c \in \mathbb{R}^d$ such that $cf(\vec{x}) = f\left(\vec{x} + \vec{t}_c\right)$ for every $\vec{x} \in \mathbb{R}^d$. Then $\vec{t}_c \in T'$ for every $c \in C$.

Since T is the group generated by T', every $\vec{t} \in T$ can be written as $\vec{t} = \sum_{i=1}^m n_i\vec{t}_i$ $\left(\vec{t}_i \in T', n_i \in \mathbb{Z}, i = 1, \ldots, m\right)$ where $f\left(\vec{x} + \vec{t}_i\right) = c_i f(\vec{x})$ $\left(\vec{x} \in \mathbb{R}^d,\ i = 1, \ldots, m\right)$.

From these we easily obtain

$$f\left(\vec{x} + \vec{t}\right) = c_{\vec{t}} f(\vec{x}), \text{ where } c_{\vec{t}} = \prod_{i=1}^{m} c_i^{n_i},\ \vec{x} \in \mathbb{R}^d,\ \vec{t} \in T. \tag{4.5}$$

Note that $c_{\vec{t}} > 0$ (and also that it is not necessarily a member of C). It suffices to show that $c_{\vec{t}} = f\left(\vec{t}\right)$ for every $\vec{t} \in T$, but this follows if we substitute $\vec{x} = \vec{0}$ into (4.5).

Since f is not identically zero, $\vec{t}_c \neq \vec{t}_{c'}$ whenever $c, c' \in C$ are distinct. Hence $\left\{\vec{t}_c : c \in C\right\}$ is uncountable, so T' is uncountable if so is C. □

Now we are ready to prove our third theorem.

Theorem 4.10. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a continuous vertically rigid function for an uncountable set $C \subseteq (0, \infty)$ via horizontal translations. Then f is of the form be^{kx} $(b \in \mathbb{R}, k \in \mathbb{R} \setminus \{0\})$.*

Proof. If f is identically zero then we are done, so let us assume that this is not the case. The class of continuous vertically rigid functions for some uncountable set via horizontal translations, as well as the class of functions of the form be^{kx} ($b \in \mathbb{R}, k \in \mathbb{R} \setminus \{0\}$) are both closed under horizontal translations and under multiplication by nonzero constants, so we may assume that $f(0) = 1$. Then Lemma 4.9 yields that $f(t_1 + t_2) = f(t_1)f(t_2)$ ($t_1, t_2 \in T$), and also that $f|_T > 0$. Then $g(t) = \log f(t)$ is defined for every $t \in T$, and g is clearly additive on T. But it is well-known (and an easy calculation) that an additive function on a dense subgroup is either of the form kx, or unbounded both from above and below on every non-degenerate interval. The second alternative cannot hold, since f is continuous, so $f|_T$ is of the form e^{kx}, so by continuity f is of this form everywhere. Since C contains elements different from 1, we obtain that $f(x) \equiv 1$ is not vertically rigid for C via horizontal translations, hence $k \neq 0$. □

Putting together the three above theorems completes the proof of Janković's conjecture.

We remark here that we have actually also proved the following, which applies e.g. to Baire class 1 functions.

Theorem 4.11. *Let $f : \mathbb{R} \to \mathbb{R}$ be a vertically rigid function for an uncountable set $C \subseteq (0, \infty)$ via translations. If f has a point of continuity then it is of the form $a + be^{kx}$ ($a, b, k \in \mathbb{R}$). If f is vertically rigid via translations (i.e. $C = (0, \infty)$) and bounded on a non-degenerate interval then it is of the form $a + be^{kx}$ ($a, b, k \in \mathbb{R}$), too.*

Proof. Following the proof of the last theorem we may assume in both cases that $f(0) = 1$, the translations are horizontal, and $f|_T$ is of the form e^{kx} ($k \in \mathbb{R}$).

In the first case, let x_0 be a point of continuity of f, then clearly $f(x_0) = e^{kx_0}$, since T is dense. Let now $x \in \mathbb{R}$ be arbitrary, and $t_n \in T$ ($n \in \mathbb{N}$) be such that $\lim_{n\to\infty} t_n = x_0 - x$. Using Lemma 4.9 we obtain

$$\begin{aligned} e^{kx_0} &= f(x_0) = \lim_{n\to\infty} f(x + t_n) = \lim_{n\to\infty} f(x)f(t_n) = f(x) \lim_{n\to\infty} f(t_n) \\ &= f(x) \lim_{n\to\infty} e^{kt_n} = f(x)e^{k(x_0 - x)} = f(x)e^{kx_0}/e^{kx}, \end{aligned}$$

from which $f(x) = e^{kx}$ follows.

In the second case, for every $c > 0$ there is a $t_c \in T = T_{f,(0,\infty)}$ such that $cf(x) = f(x+t_c) = f(x)f(t_c)$. By substituting $x = 0$ into the equation we obtain $c = f(t_c) = e^{kt_c}$ for every $c > 0$. (In particular, $k \neq 0$.) So $t_c = \frac{\log c}{k}$. If c ranges over $(0, \infty)$ then t_c ranges over $\mathbb{R}$, so we obtain $T = \mathbb{R}$. Hence $f|_T = f$ is of the form e^{kx}, and we are done. □

Example 4.12. There exists a function $f : \mathbb{R} \to \mathbb{R}$ that is vertically rigid for an uncountable set $C \subseteq \mathbb{R}$ via horizontal translations, bounded on every bounded interval, and is *not* of the form $a + be^{kx}$ ($a, b, k \in \mathbb{R}$).

Proof. Let $P \subseteq \mathbb{R}$ be an uncountable linearly independent set over $\mathbb{Q}$, see e.g. [23, 19.2] or [31]. Define $\widehat{P}$ to be the generated additive subgroup. Let

$$f(x) = \begin{cases} e^x & \text{if } x \in \widehat{P} \\ 0 & \text{if } x \in \mathbb{R} \setminus \widehat{P}, \end{cases}$$

then f is clearly bounded on every bounded interval.

It is easy to see that $\frac{p}{2} \in \mathbb{R} \setminus \widehat{P}$ for every $p \in P$, so $\widehat{P} \neq \mathbb{R}$, hence f is not continuous, so it is not of the form $a + be^{kx}$ $(a, b, k \in \mathbb{R})$.

For every $p \in P$ and $x \in \mathbb{R}$ we have $x \in \widehat{P} \iff x + p \in \widehat{P}$, which easily implies $f(x + p) = e^p f(x)$. Hence f is vertically rigid for the uncountable set $C = \{e^p : p \in P\}$. $\square$

Janković's conjecture has the following curious corollary.

Corollary 4.13. *There are continuous functions f and g with isometric graphs so that f is vertically rigid but g is not.*

Proof. If we rotate the graph of $f(x) = e^x$ clockwise by $\frac{\pi}{4}$ then we obtain the graph of a continuous function. By Theorem 4.4 it is not vertically rigid. $\square$

4.1.3 A Borel measurable counterexample

In this subsection we show that Janković's conjecture fails for Borel measurable functions. Our example also answers Question 1 in [8] of Cain, Clark and Rose, which asks whether every vertically rigid function is of the form $a + bx$ $(a, b \in \mathbb{R})$ or $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g. By Thm. 2 of [8] $a + be^g$ is vertically rigid if and only if $b = 0$ or g is surjective.

Theorem 4.14. *There exists a Borel measurable vertically rigid function $f : \mathbb{R} \to [0, \infty)$ (via horizontal translations) that is not of the form $a + bx$ $(a, b \in \mathbb{R})$ or $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g.*

For definitions and basic results on Baire measurable sets (= sets with the property of Baire), meager (= first category) and co-meager (= residual) sets consult e.g. [32] or [23]. For Polish spaces and Borel isomorphisms see e.g. [23].

Proof. Let P be a Cantor set (nonempty nowhere dense compact set with no isolated points) that is linearly independent over $\mathbb{Q}$, see e.g. [23, 19.2]. (One can also derive the existence of such a set from [31] using the well-known fact that every uncountable Borel or analytic set contains a Cantor set.) It is easy to see that for all $n_1, \dots, n_k \in \mathbb{Z}$ the set $P_{n_1,\dots,n_k} = \{n_1p_1 + \dots + n_kp_k : p_1, \dots, p_k \in P\}$ is compact, hence the group $\widehat{P}$ generated by P (that is, the union of the $P_{n_1,\dots,n_k}$'s) is a Borel, actually F_σ set. As P is linearly independent, each element of $\widehat{P}$ can be uniquely written in the form $n_1p_1 + \dots + n_kp_k$.

Since P and $(0, \infty)$ are uncountable Polish spaces, we can choose a Borel isomorphism $g: P \to (0, \infty)$. Let $f: \mathbb{R} \to \mathbb{R}$ be the following function:

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \widehat{P} \\ \prod_{i=1}^{k} g(p_i)^{n_i} & \text{if } x = \sum_{i=1}^{k} n_ip_i \in \widehat{P},\ n_i \in \mathbb{Z},\ p_i \in P,\ i = 1, \dots, k. \end{cases}$$

This function is Borel, as it is Borel on the countably many Borel sets $P_{n_1,\dots,n_k}$, and zero on the rest. However, f is not continuous, as it is unbounded on the compact set P. Therefore f is not of the form $a + bx$. Suppose now that f is of the form $a + be^g$ for some $a, b \in \mathbb{R}$ and additive function g. Clearly $b \neq 0$, since f is not constant, therefore $\frac{f-a}{b} = e^g$ is Borel measurable, and then so is g by taking logarithm. But it is well-known that every Borel (or even Lebesgue) measurable additive function is of form kx ($k \in \mathbb{R}$), hence f is continuous, a contradiction.

What remains to show is that f is vertically rigid via horizontal translations. For every $c > 0$ there exists a $p \in P$ such that $g(p) = c$. Now we check that $cf(x) = f(x+p)$ for all $x \in \mathbb{R}$. Clearly $x \in \widehat{P}$ if and only if $x + p \in \widehat{P}$. Therefore $cf(x) = f(x + p) = 0$ if $x \notin \widehat{P}$. Let now $x = n_1p_1 + \dots + n_kp_k \in \widehat{P}$, and assume without loss of generality that $p = p_1$ ($n_1 = 0$ is also allowed). Then $cf(x) = g(p)f(x) = g(p)g(p)^{n_1}g(p_2)^{n_2} \cdots g(p_k)^{n_k} = g(p)^{n_1+1}g(p_2)^{n_2} \cdots g(p_k)^{n_k} = f((n_1 + 1)p + n_2p_2 + \dots + n_kp_k) = f(x+p)$, which finishes the proof. □

4.1.4 Lebesgue and Baire measurable functions

It is easy to see that the example in the previous subsection is zero almost everywhere (on a co-meager set). Indeed, it can be shown that every $P_{n_1,\dots,n_k}$ has uncountably many pairwise disjoint translates.

Therefore it is still possible that the complete analogue of Janković's conjecture holds: every vertically rigid Lebesgue (Baire) measurable function is of the form $a + bx$ or $a + be^{kx}$ *almost everywhere (on a co-meager set)*. In this subsection we prove this in case of translations. The general case remains open, see Section 4.4. We also prove that in many situations the exceptional set can be removed.

Theorem 4.15. *Let $f: \mathbb{R} \to \mathbb{R}$ be a vertically rigid function for an uncountable set $C \subseteq (0, \infty)$ via translations. If f is Lebesgue (Baire) measurable then it is of the form*

$a + be^{kx}$ $(a, b, k \in \mathbb{R})$ *almost everywhere (on a co-meager set).*

Proof. By Theorem 4.7 we can assume that f is vertically rigid for C via horizontal translations. As in the proof of Theorem 4.10 we can also assume that $f(0) = 1$. Then Lemma 4.9 implies that

$$f(x+t) = f(x)f(t) \quad \forall x \in \mathbb{R}\ \forall t \in T \tag{4.6}$$

and $f(t) > 0$ for every $t \in T$.

First we show that the sign of f is constant almost everywhere (on a co-meager set). It is easy to see from (4.6) that the sets $\{f > 0\}$, $\{f = 0\}$, and $\{f < 0\}$ are all Lebesgue (Baire) measurable sets periodic modulo every $t \in T$. It is a well-known and easy consequence of the Lebesgue density theorem (the fact that every set with the Baire property is open modulo meager) that if a measurable set H has a dense set of periods then either H or $\mathbb{R} \setminus H$ is of measure zero (meager). But the above three sets cover $\mathbb{R}$, hence at least one of them is of positive measure (non-meager), and then that one is of full measure (co-meager). If $f = 0$ almost everywhere (on a co-meager set) then we are done, otherwise we may assume that $f > 0$ almost everywhere (on a co-meager set). (Indeed, $-f$ is also rigid via horizontal translations, and then we can apply a horizontal translation and a positive multiplication to achieve $f(0) = 1$.)

Set $D = \{f > 0\}$ and define the measurable function $g = \log f$ on D. Recall that $D + t = D$ $(\forall t \in T)$ and note that $T \subseteq D$. Clearly

$$g(x+t) = g(x) + g(t) \quad \forall x \in D\ \forall t \in T,$$

so $g|_T$ is additive. Now we show that $g|_T$ is of the form kx. Let us suppose that this is not the case. As we have mentioned above, if an additive function is not of the form kx then it is unbounded on every interval from above (and also below). For every Lebesgue (Baire) measurable function there is a measurable set of positive measure (non-meager) on which the function is bounded. So let $M \subseteq D$ be a measurable set of positive measure (non-meager) such that $|g|_M| \le K$ for some $K \in \mathbb{R}$. By the Lebesgue density theorem (the fact that every Baire measurable set is open modulo meager) there exists $\varepsilon > 0$ so that $(M+s) \cap M \neq \emptyset$ for every $s \in (-\varepsilon, \varepsilon)$. Choose $t_0 \in T$ in $(-\varepsilon, \varepsilon)$ so that $g(t_0) > 2K$. Fix an arbitrary $m_0 \in M \cap (M - t_0)$, then $g(m_0 + t_0) = g(m_0) + g(t_0) > g(m_0) + 2K$, which is absurd, since $m_0 + t_0, m_0 \in M$ and $|g|_M| \le K$.

Now define $h(x) = g(x) - kx$ $(x \in D)$. This is a measurable function that is periodic modulo every $t \in T$. Indeed,

$$h(x+t) = g(x+t) - k(x+t) = g(x) - kx + g(t) - kt = h(x) + 0 = h(x).$$

It is a well-known consequence of the Lebesgue density theorem (the fact that every Baire measurable set is open modulo meager) that if the periods of a measurable function form a dense set then the function is constant almost everywhere (on a co-meager set). Hence $g(x) = kx + c$ almost everywhere (on a co-meager set), so $f(x) = e^c e^{kx}$ almost everywhere (on a co-meager set), so we are done. $\square$

Our next theorem shows that the measure zero (meager) set can be removed, unless f is constant almost everywhere (on a co-meager set). Theorem 4.14 provides an almost everywhere (on a co-meager set) constant but nonconstant function that is vertically rigid via horizontal translations.

Theorem 4.16. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a vertically rigid function that is of the form $a + bx$ ($b \neq 0$) or $a + be^{kx}$ ($bk \neq 0$) almost everywhere (on a co-meager set). Then f is of this form everywhere.*

Let us denote the one-dimensional Hausdorff measure by $\mathcal{H}^1$. For the definition and properties see [13] or [29]. First we prove the following lemma.

Lemma 4.17. *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be arbitrary functions, and let φ be an isometry such that $\varphi(\mathrm{graph}(f)) = \mathrm{graph}(g)$. Let $f', g'\colon \mathbb{R} \to \mathbb{R}$ be continuous functions such that $f' = f$ almost everywhere (on a co-meager set) and $g' = g$ almost everywhere (on a co-meager set). Let us also assume that $\mathrm{graph}(f')$, $\varphi(\mathrm{graph}(f'))$, $\mathrm{graph}(g')$ and $\varphi^{-1}(\mathrm{graph}(g'))$ are coverable by the graphs of countably many Lipschitz (continuity suffices for the category case) functions. Then $\varphi(\mathrm{graph}(f')) = \mathrm{graph}(g')$.*

Proof. By symmetry of f' and g' (with φ^{-1}), it suffices to show that $\mathrm{graph}(g') \subseteq \varphi(\mathrm{graph}(f'))$. Since the latter set is closed, it also suffices to show that $\varphi(\mathrm{graph}(f'))$ covers a dense subset of $\mathrm{graph}(g')$. We will actually show that $\varphi(\mathrm{graph}(f'))$ covers $\mathcal{H}^1$ a.e. (relatively co-meager many) points of $\mathrm{graph}(g')$, which will finish the proof.

If an element of $\mathrm{graph}(g')$ fails to be covered by $\varphi(\mathrm{graph}(f'))$ then it is either in $\mathrm{graph}(g') \setminus \mathrm{graph}(g)$ or in $\varphi(\mathrm{graph}(f) \setminus \mathrm{graph}(f')) \cap \mathrm{graph}(g')$. The first set is clearly of $\mathcal{H}^1$ measure zero (relatively meager in $\mathrm{graph}(g')$), so it suffices to show that this is also true for the second. Equivalently, we need that $\mathrm{graph}(f) \setminus \mathrm{graph}(f')$ only covers a $\mathcal{H}^1$ measure zero (relatively meager) subset of $\varphi^{-1}(\mathrm{graph}(g'))$. Suppose that $\varphi^{-1}(\mathrm{graph}(g')) \subseteq \bigcup_{n=1}^{\infty} \mathrm{graph}(h_n)$, where the h_n's are Lipschitz (continuous) functions. As $\mathrm{graph}(h_n) \cap (\mathrm{graph}(f) \setminus \mathrm{graph}(f'))$ is clearly of $\mathcal{H}^1$ measure zero for every n, we are done in the measure case.

Let us now write $\{x \in \mathbb{R} : f'(x) \neq f(x)\} = \bigcup_{m=1}^{\infty} N_m$, where each N_m is nowhere dense. It is enough to show that each $\mathrm{graph}(f|_{N_m})$ only covers a relatively nowhere dense subset of $\varphi^{-1}(\text{graph}(g'))$. Fix an m, and suppose that $\text{graph}(f|_{N_m})$ is dense in an open subarc $U \subseteq \varphi^{-1}(\text{graph}(g'))$. By the Baire Category Theorem there exists a relatively open subarc $V \subseteq U$ that is covered by one of the $\text{graph}(h_n)$'s. But this is impossible, as the arc V is in $\text{graph}(h_n)$, and the set $N_m \subseteq \mathbb{R}$ is nowhere dense, so even $N_m \times \mathbb{R}$ covers at most a relatively nowhere dense subset of V, hence $\text{graph}(f|_{N_m})$ cannot be dense in V. □

Proof of Theorem 4.16. Using the notation of the above lemma, let first f be a vertically rigid function such that $f = f'$ almost everywhere (on a co-meager set), where f' is of the form $a + be^{kx}$ ($bk \neq 0$). The above lemma implies that f' is also vertically rigid with the same isometries φ_c. By considering the unique asymptote and the limit at $\pm\infty$ of f' we obtain that every φ_c is a translation. By Theorem 4.7 we may assume that every φ_c is actually horizontal, hence f' is of the form be^{kx}. Hence $cf'(x) = f'\left(x + \frac{\log(c)}{k}\right)$ for every $x \in \mathbb{R}$, $c > 0$ and the same holds for f. Assume now that there is an x_0 so that $f(x_0) \neq f'(x_0)$, then $cf(x_0) \neq cf'(x_0)$ for every $c > 0$, therefore $f\left(x_0 + \frac{\log(c)}{k}\right) \neq f'\left(x_0 + \frac{\log(c)}{k}\right)$ for every $c > 0$, which is a contradiction as $f = f'$ almost everywhere (on a co-meager set).

Assume now that f' is of the form $a + bx$ ($b \neq 0$). First we show that f' is vertically rigid by the same isometries as f. For every $c > 0$ set $g = cf$, $g' = cf'$, and let φ_c be the isometry mapping $\text{graph}(f)$ onto $\text{graph}(g)$. As $\text{graph}(f) \cap \text{graph}(f')$ contains at least two points and $\varphi_c(\text{graph}(f) \cap \text{graph}(f'))$ is the graph of a function we obtain that the line $\varphi_c(\text{graph}(f'))$ is not vertical, and similarly for $\varphi_c^{-1}(\text{graph}(g'))$. Therefore they are coverable by the graphs of countably many, actually a single, Lipschitz (continuous) function, hence the previous lemma applies. Hence f' is vertically rigid by the same isometries as f.

Similarly to Theorem 4.6 we can assume that f is vertically rigid via orientation preserving isometries for a set C of positive outer measure (non-meager). So φ_c is a rotation or translation for every $c \in C$, and by splitting C into two parts and keeping one with positive outer measure (non-meager), we can assume that $A = \{\text{ang}(\varphi_c) : c \in C\}$ is a subset of the left or the right half of the unit circle. We could calculate $\text{ang}(\varphi_c)$ explicitly, but we only need that it is a nonconstant real analytic function. From this it is easy to see that the set A is of positive outer measure (non-meager). Assume now that there is an x_0 so that $f(x_0) \neq f'(x_0)$. We prove that this contradicts the fact that $\varphi_c(\text{graph}(f))$ is the graph of a function for every $c \in C$. For this it suffices to show that S_f (see Theorem 4.6) is of full measure (co-meager). But this clearly follows simply by looking at the pairs (p_0, q) and (q, p_0) where $p_0 = (x_0, f(x_0))$ and q ranges over $\text{graph}(f) \cap \text{graph}(f')$. $\square$

In Subsection 4.1.3 we have constructed a nonconstant Borel measurable function $f\colon \mathbb{R} \to \mathbb{R}$ that is vertically rigid *via translations*, and equals zero almost everywhere (on a co-meager set). By adding a constant to f we can also say constant instead of zero. Thus the following theorem is sharp in a sense.

Theorem 4.18. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a vertically rigid function* via translations. *If f is Lebesgue (Baire) measurable, then it is constant almost everywhere (on a co-meager set), or it is of the form $a + be^{kx}$ $(bk \neq 0)$ everywhere.*

Proof. It follows easily from Theorem 4.15 and Theorem 4.16. $\square$

4.1.5 Rigid sets

The starting point is the proof of Theorem 4.6. So far we are only able to prove this result for continuous functions, and consequently we can only handle translations in the Borel/Lebesgue/Baire measurable case. But generalizations of the ideas concerning the sets S_f could tackle this difficulty. For a Borel function f the set S_f is analytic (see e.g. [23]), and every analytic set has the Baire property, so the result of this subsection can be considered as the first step towards handling Borel functions with general isometries. See equation (4.1) for the following notations.

Definition 4.19. We call a symmetric (about the origin) set $H \subseteq \mathbb{S}^1$ *rigid* for a set $C \subseteq (0, \infty)$ if for every $c \in C$ there is an α such that

$$H = \varrho_\alpha \left(\psi_c(H)\right). \tag{4.7}$$

Lemma 4.20. *Let U be a regular open set (i.e. $\operatorname{int}(\operatorname{cl}(U)) = U$) that is rigid for an uncountable set C. Then $U = \emptyset$, or $U = \mathbb{S}^1$, or every connected component of U is an interval whose endpoints are among $(0, \pm 1)$ and $(\pm 1, 0)$.*

Proof. Let A be the set of arclengths of the connected components of U, then A is countable. Let I be a connected component of U showing that U is not of the desired form, then $0 < \text{arclength}(I) < \pi$ since U is symmetric and regular. As in the proof of Theorem 4.6 let us prove that the real analytic function $l(c) = \text{arclength}\left(\psi_c(I)\right)$ $(c \in (0, \infty))$ is not constant. If I is in the left or right half of $\mathbb{S}^1$ then we already showed this there, so we may assume that $(0, 1)$ or $(0, -1)$ is in I. Since $\lim_{c \to \infty} \psi_c(x) \in \{(0, \pm 1), (\pm 1, 0)\}$ for every $x \in \mathbb{S}^1$, we obtain that $\lim_{c \to \infty} l(c) \in \mathbb{Z}\frac{\pi}{2}$. Hence we are done using $0 < \text{arclength}(I) < \pi$ unless $\text{arclength}(I) = \frac{\pi}{2}$. But if $\text{arclength}(I) = \frac{\pi}{2}$ then $\lim_{c\to\infty} l(c) = 0$ since $(0,1)$ or $(0,-1)$ is in I, and therefore l cannot be constant.

Hence l attains each of its values at most countably many times, so there is a $c \in C$ such that $\text{arclength}\,(\psi_c(I)) \notin A$, contradicting (4.7). ☐

One can also show, using an argument similar to the above one (by considering the possible distances of pairs in H), that the rigid sets (for $C = (0,\infty)$) of cardinality smaller than the continuum are the following: the empty set, the symmetric sets of two elements and the set $\{(0,\pm1),(\pm1,0)\}$.

The next statement is somewhat of ergodic theoretic flavour.

Theorem 4.21. *Let H be a Baire measurable set that is rigid for an uncountable set C. Then in each of the four quarters of $\mathbb{S}^1$ determined by $(0,\pm1)$ and $(\pm1,0)$ either H or $\mathbb{S}^1 \setminus H$ is meager.*

Proof. H can be written as $H = U \Delta F$ in a unique way, where U is regular open, F is meager and Δ stands for symmetric difference, see [32, 4.6]. Then it is easy to see by the uniqueness of U that U is rigid for C, so we are done by Lemma 4.20. ☐

4.2 The structure of continuous rigid functions of two variables

4.2.1 Outline

The aim of the section is to prove the following theorem.

Theorem 4.22. *A continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid if and only if after a suitable rotation around the z-axis $f(x,y)$ is of the form $a + bx + dy$, $a + s(y)e^{kx}$ or $a + be^{kx} + dy$ $(a,b,d,k \in \mathbb{R},\ k \neq 0,\ s\colon \mathbb{R} \to \mathbb{R}$ continuous).*

We prove a stronger technical form of Theorem 4.22, so we need the following definition.

Definition 4.23. Let us say that a set $C \subseteq (0,\infty)$ *condensates to* ∞ if for every $r \in \mathbb{R}$ the set $C \cap (r,\infty)$ is uncountable.

Theorem 4.24 (Theorem 4.22, technical form)**.** *Let $C \subseteq (0,\infty)$ be a set condensating to ∞. Then a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for C if and only if after a suitable rotation around the z-axis $f(x,y)$ is of the form $a+bx+dy$, $a+s(y)e^{kx}$ or $a + be^{kx} + dy$ $(a,b,d,k \in \mathbb{R},\ k \neq 0,\ s\colon \mathbb{R} \to \mathbb{R}$ continuous).*

We follow the strategy described in Section 1.4.

4.2.2 The functions of Theorem 4.24 are vertically rigid

Rotation of the graph around the z-axis does not affect vertical rigidity, so we can assume that f is of the given form without rotations.

Functions of the form $a + bx + dy$ are clearly vertically rigid.

Let now $f(x, y) = a + s(y)e^{kx}$ ($a, k \in \mathbb{R}$, $k \neq 0$, $s\colon \mathbb{R} \to \mathbb{R}$ continuous). Then $cf(x, y) = f\left(x + \frac{\log c}{k}, y\right) + a(c - 1)$, so f is actually vertically rigid *via translations* in the xz-plane.

Before checking the third case we need a lemma.

Lemma 4.25. *Let $f(x, y) = g(x)+dy$, where $d > 0$ and let $c > 0$. If we rotate* $\operatorname{graph}(f)$ *around the x-axis by the angle $\alpha_c = \arctan(cd) - \arctan(d)$ then the intersection of this rotated graph with the xy-plane is the graph of a function of the form $y = -w_{c,d}g(x)$, where $w_{c,d} > 0$ and the map $c \mapsto w_{c,d}$ is strictly monotone on $(0, \infty)$ for every fixed $d > 0$.*

Remark 4.26. By rather easy and short elementary geometric considerations one can check that for every fixed $d > 0$ the map $c \mapsto w_{c,d}$ is positive and real analytic. It is also very easy to see geometrically that the limit at 0 is ∞, hence it is not constant, therefore countable-to-one. This would suffice for all our purposes, but these arguments are unfortunately very hard to write down rigorously, so we decided to present a less instructive and longer algebraic proof.

Proof. Using the matrix of the rotation we can write the rotated image of the point of the graph $(x, y_0, g(x) + dy_0)$ as

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha_c & -\sin\alpha_c \\ 0 & \sin\alpha_c & \cos\alpha_c \end{pmatrix} \begin{pmatrix} x \\ y_0 \\ g(x) + dy_0 \end{pmatrix} = \begin{pmatrix} x \\ y_0(\cos\alpha_c - d\sin\alpha_c) - g(x)\sin\alpha_c \\ y_0(\sin\alpha_c + d\cos\alpha_c) + g(x)\cos\alpha_c \end{pmatrix}. \tag{4.8}$$

Let us now determine the intersection of the rotated graph with the xy-plane. This right hand side of (4.8) is in the xy-plane if and only if the third coordinate vanishes, that is, when $y_0(\sin\alpha_c + d\cos\alpha_c) + g(x)\cos\alpha_c = 0$. This yields

$$y_0 = -\frac{\cos\alpha_c}{\sin\alpha_c + d\cos\alpha_c} g(x). \tag{4.9}$$

In order to complete the proof of the lemma we have to calculate the y-coordinate of the rotated image of the point $(x, y_0, g(x) + dy_0)$, which is the second entry of the right hand side of (4.8). Hence, using (4.9),

$$\begin{aligned}
y &= y_0(\cos\alpha_c - d\sin\alpha_c) - g(x)\sin\alpha_c \\
&= -\frac{\cos\alpha_c(\cos\alpha_c - d\sin\alpha_c)}{\sin\alpha_c + d\cos\alpha_c} g(x) - g(x)\sin\alpha_c \\
&= -\frac{\cos^2\alpha_c - d\cos\alpha_c\sin\alpha_c + \sin^2\alpha_c + d\cos\alpha_c\sin\alpha_c}{\sin\alpha_c + d\cos\alpha_c} g(x) \\
&= -\frac{1}{\sin\alpha_c + d\cos\alpha_c} g(x).
\end{aligned}$$

Therefore

$$w_{c,d} = \frac{1}{\sin\alpha_c + d\cos\alpha_c} = \left(\sqrt{d^2+1}\left(\sin\alpha_c \frac{1}{\sqrt{d^2+1}} + \cos\alpha_c \frac{d}{\sqrt{d^2+1}}\right)\right)^{-1}.$$

Using the identity

$$\sin\alpha = \frac{\tan\alpha}{\sqrt{\tan^2\alpha + 1}} \quad (\alpha \in (-\pi/2, \pi/2)) \tag{4.10}$$

we obtain $\sin(\arctan(d)) = \frac{d}{\sqrt{d^2+1}}$, which easily implies $\cos(\arctan(d)) = \frac{1}{\sqrt{d^2+1}}$. (Note that $\arctan(d) \in (-\pi/2, \pi/2)$.) So

$$w_{c,d} = \left(\sqrt{d^2+1}\big(\sin\alpha_c\cos(\arctan(d)) + \cos\alpha_c\sin(\arctan(d))\big)\right)^{-1}.$$

By the formula $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$ and the definition of α_c this equals

$$\left(\sqrt{d^2+1}\sin(\alpha_c + \arctan(d))\right)^{-1} = \left(\sqrt{d^2+1}\sin(\arctan(cd))\right)^{-1}.$$

Applying (4.10) again yields

$$w_{c,d} = \left(\sqrt{d^2+1}\frac{\tan(\arctan(cd))}{\sqrt{\tan^2(\arctan(cd))+1}}\right)^{-1} = \sqrt{\frac{1}{d^2+1}\left(1 + \frac{1}{(cd)^2}\right)}.$$

From this form it is easy to see that this function is positive and strictly monotone on $(0,\infty)$ for every fixed $d > 0$. □

Let now $f(x,y) = a + be^{kx} + dy$ $(a,b,d,k \in \mathbb{R},\ k \neq 0)$. Rescaling the graph in a homothetic way does not affect vertical rigidity, so we can consider $kf(\frac{x}{k}, \frac{y}{k})$ and assume $k = 1$. We may also assume $b, d \neq 0$, otherwise our function is of one of the previous forms. Adding a constant, reflecting the graph about the xz-plane (needed only if the signs of b and d differ), multiplying by a nonzero constant, as well as a translation in the x-direction do not affect vertical rigidity, so by applying these in this order we can assume that $a = 0$, $bd > 0$, $d = 1$, and $b = 1$.

Hence it suffices to check that $f(x,y) = e^x + y$ is vertically rigid. Let us fix a $c > 0$. In every vertical plane of the form $\{x = x_0\}$ the restriction of f is a straight line of slope 1. Rotation around the x-axis by angle $\alpha_c = \arctan(c) - \frac{\pi}{4}$ takes all these lines to lines of slope c. By applying Lemma 4.25 with $g(x) = e^x$ and $d = 1$, the intersection of the rotated graph and the xy-plane is the graph of the function $y = -w_{c,1}e^x$.

Now, applying a translation in the x-direction we can obtain a function with still all lines of slope c but now with intersection with the xy-plane of the form $y = -e^x$ (note that $w_{c,1} > 0$). But then we are done, since this function clearly agrees with cf. (The intersection of graph(f) and the xy-plane is of the form $y = -e^x$, and all lines in this graph are of slope 1, hence for graph(cf) the intersection is still $y = -e^x$, and all lines are of slope c.) This finishes the proof of vertical rigidity.

4.2.3 Vertical rigidity via translations

The following lemma will be useful throughout the section. Sometimes we will use it tacitly. The easy proof is left to the reader.

Lemma 4.27. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be vertically rigid for c_0 via φ_0 and for c via φ. Then $c_0 f$ is vertically rigid for $\frac{c}{c_0}$ via $\varphi \circ \varphi_0^{-1}$.*

Theorem 4.28. *Let $C \subseteq (0,\infty)$ be an uncountable set. Then a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for C* via horizontal translations *if and only if after a suitable rotation around the z-axis $f(x,y)$ is of the form $s(y)e^{kx}$ ($k \in \mathbb{R}$, $k \neq 0$, $s\colon \mathbb{R} \to \mathbb{R}$ continuous).*

Proof. We already checked the easy direction in the previous subsection, it is enough to prove the other direction. If f is identically zero then we are done, so let us assume that this is not the case. The class of continuous vertically rigid functions for some set condensating to ∞ via horizontal translations, as well as the class of functions of the form $s(y)e^{kx}$ ($k \in \mathbb{R}$, $k \neq 0$, $s\colon \mathbb{R} \to \mathbb{R}$ continuous) are both closed under horizontal translations and under multiplication by nonzero constants (by Lemma 4.27). Hence we may assume that $f(\vec{0}) = 1$. Then Lemma 4.9 yields that $f\left(\vec{t}_1 + \vec{t}_2\right) = f\left(\vec{t}_1\right) f\left(\vec{t}_2\right)$ $\left(\vec{t}_1, \vec{t}_2 \in T\right)$, and also that $f|_T > 0$, for the definition of T see Definition 4.8. Then $g\left(\vec{t}\right) = \log f\left(\vec{t}\right)$ is defined for every $\vec{t} \in T$, and g is clearly additive on T.

Let us now consider $\bar{T}$, the closure of T, which is clearly an uncountable closed subgroup of $\mathbb{R}^2$. It is well-known that every closed subgroup of $\mathbb{R}^2$ is a non-degenerate linear image of a group of the form $G_1 \times G_2$, where $G_1, G_2 \in \{\{0\}, \mathbb{Z}, \mathbb{R}\}$. Hence after a suitable rotation around the origin $\bar{T}$ is either $\mathbb{R}^2$ or $\mathbb{R} \times \{0\}$ or $\mathbb{R} \times r\mathbb{Z}$ for some $r > 0$.

Case 1. $\bar{T} = \mathbb{R}^2$.

In this case $T \subseteq \mathbb{R}^2$ is dense. It is well-known that a continuous additive function on a dense subgroup is of the form $g(x, y) = \alpha x + \beta y$, $((x, y) \in T)$ for some $\alpha, \beta \in \mathbb{R}$. But then $f(x, y) = e^{\alpha x + \beta y}$ on T, and by continuity this holds on the whole plane as well. As the constant 1 function is not vertically rigid via horizontal translations, $\alpha = \beta = 0$ cannot hold. By applying a rotation of angle $\frac{\pi}{2}$ if necessary we may assume that $\alpha \neq 0$. But then by choosing $k = \alpha$, $s(y) = e^{\beta y}$ we are done.

Case 2. $\bar{T} = \mathbb{R} \times \{0\}$.

In this case every $\vec{t_c}$ is of the form $(t_c, 0)$, where $t_c \neq 0$ if $c \neq 1$. (We may assume $1 \notin C$.)

Applying Theorem 4.10 for every fixed y we get that $f(x, y) = s(y)e^{k_y x}$ $(s(y), k_y \in \mathbb{R}, k_y \neq 0)$. As $s(y) = f(0, y)$, we obtain that s is continuous. If $s(y) \neq 0$ then it is not hard to see that $k_y = \frac{\log c}{t_c}$, which is independent of y, so for these y's $k_y = k$ is constant. But if $s(y) = 0$ then the value of k_y is irrelevant, so it can be chosen to be the same constant k. Hence without loss of generality $k_y = k$ is constant, and we are done with this case.

Case 3. $\bar{T} = \mathbb{R} \times r\mathbb{Z}$.

As T' is uncountable, there is an $n \in \mathbb{Z}$ so that $T' \cap (\mathbb{R} \times \{rn\})$ is uncountable. Fix an element t_{c_0} of this set. Then Lemma 4.27 yields that $c_0 f$ is vertically rigid for an uncountable set *via translations* of the form $(t, 0)$. Restricting ourselves to these isometries and c's we are done using Case 2, since every uncountable set in $\mathbb{R}$ generates a dense subgroup. $\square$

Now we handle the case of arbitrary translations. Theorem 4.28 and Theorem 4.7 easily implies the following.

Corollary 4.29. *Let $C \subseteq (0, \infty)$ be an uncountable set. Then a continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is vertically rigid for C* via translations *if and only if after a suitable rotation around the z-axis $f(x, y)$ is of the form $a + s(y)e^{kx}$ $(a, k \in \mathbb{R},\ k \neq 0,\ s\colon \mathbb{R} \to \mathbb{R}$ continuous).*

4.2.4 The set S_f

Now we start working on the case of arbitrary isometries.

Let $\mathbb{S}^2 \subseteq \mathbb{R}^3$ denote the unit sphere. For a function $f\colon \mathbb{R}^2 \to \mathbb{R}$ let S_f be the set of directions between pairs of points on the graph of f, that is,

Definition 4.30.

$$S_f = \left\{ \frac{p-q}{|p-q|} \in \mathbb{S}^2 : p, q \in \mathrm{graph}(f),\ p \neq q \right\}.$$

Recall that a *great circle* is a circle line in $\mathbb{R}^3$ of radius 1 centered at the origin. We call it *vertical* if it passes through the points $(0, 0, \pm 1)$.

Lemma 4.31. *Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be continuous. Then*

1. $-S_f = S_f$ *(symmetric about the origin),*
2. $(0, 0, \pm 1) \notin S_f$*,*
3. S_f *is connected,*
4. *every great circle containing $(0, 0, \pm 1)$ intersects S_f in two (symmetric) nonempty arcs,*
5. $\mathbb{S}^2 \setminus S_f$ *has exactly two connected components, one containing $(0, 0, 1)$ and one containing $(0, 0, -1)$.*

Proof. (1.) Obvious.

(2.) Obvious, since f is a function.

(3.) $\mathrm{graph}(f)$ is homeomorphic to $\mathbb{R}^2$, so the squared of it minus the (2-dimensional) diagonal is a connected set. Since S_f is the continuous image of this connected set, it is itself connected.

(4.) The intersection of S_f with such a great circle corresponds to restricting our attention to distinct pairs of points $(\vec{x}_1, \vec{x}_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ so that the segment $[\vec{x}_1, \vec{x}_2]$ is parallel to a fixed line $L \subseteq \mathbb{R}^2$. Now, given two such non-degenerate segments it is easy to move one of them continuously to the other so that along the way it remains non-degenerate and parallel to L. This shows that in both halves of the great circle (separated by $(0, 0, \pm 1)$) S_f is pathwise connected, hence it is an arc.

(5.) By (4.) every point of $\mathbb{S}^2 \setminus S_f$ can be connected with an arc of a vertical great circle either to $(0, 0, 1)$ or to $(0, 0, -1)$ in $\mathbb{S}^2 \setminus S_f$, hence there are at most two connected components.

Now we show that $(0, 0, 1)$ and $(0, 0, -1)$ are in different ones. It suffices to show that there exists a Jordan curve in S_f so that $(0, 0, 1)$ and $(0, 0, -1)$ are in the two distinct components of its complement. Let $\mathbb{S}^1$ denote the unit circle in $\mathbb{R}^2 = \{(x, y, z) : z = 0\}$ and let $\gamma \colon \mathbb{S}^1 \to S_f$ be given by

$$\gamma(\vec{x}) = \frac{(\vec{x}, f(\vec{x})) - (-\vec{x}, f(-\vec{x}))}{|(\vec{x}, f(\vec{x})) - (-\vec{x}, f(-\vec{x}))|}.$$

In this paragraph the word 'component' will refer to the components of $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$. One can easily check that γ is continuous and injective, hence a Jordan curve. Moreover, it is clearly in S_f, and its intersection with every vertical great circle is a symmetric pair of points. Therefore every point of $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$ can be connected with an arc of a vertical great circle either to $(0,0,1)$ or to $(0,0,-1)$ in $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$, hence the union of the components of $(0,0,1)$ and $(0,0,-1)$ cover $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$. So $(0,0,1)$ and $(0,0,-1)$ are in different components, otherwise $\mathbb{S}^2 \setminus \gamma(\mathbb{S}^1)$ would be connected, but the complement of a Jordan curve in $\mathbb{S}^2$ has two components. □

The above lemma shows that S_f is something like a 'strip around the sphere'. Now we make this somewhat more precise by defining the top and the bottom 'boundaries' of this strip.

Definition 4.32. Let $h \colon \mathbb{S}^1 \to \mathbb{S}^2$ be defined as follows. Every $\vec{x} \in \mathbb{S}^1$ is in a unique half great circle connecting $(0,0,1)$ and $(0,0,-1)$. The intersection of S_f with this great circle is an arc, define $h(\vec{x})$ as the top endpoint of this arc.

Clearly, the bottom endpoint of this arc is $-h(-\vec{x})$, so the 'top function bounding the strip S_f is $h(\vec{x})$ and the bottom function is $-h(-\vec{x})$'. The coordinate functions of h are denoted by (h_1, h_2, h_3), where $h_3 \colon \mathbb{S}^1 \to [-1,1]$ encodes all information about h.

Lemma 4.33. *Let $f \colon \mathbb{R}^2 \to \mathbb{R}$ be continuous, and h be defined as above. Then*

1. *$h(\vec{x}) \neq (0,0,-1)$ for every $\vec{x} \in \mathbb{S}^1$*

2. *h is lower semi-continuous (in the obvious sense, or equivalently, h_3 is lower semi-continuous)*

3. *h is convex with respect to great circles, that is, if $h(\vec{x})$ and $h(\vec{y})$ determine a unique non-vertical great circle (i.e. there is a subarc of $\mathbb{S}^1$ of length $< \pi$ connecting $\vec{x}$ and $\vec{y}$, and $h(\vec{x}), h(\vec{y}) \neq (0,0,1)$) then on this subarc* $\operatorname{graph}(h)$ *is bounded from above by the great circle.*

Proof. (1.) Obvious by Lemma 4.31 (2.) and (4.).

(2.) We have to check that if $h_3(\vec{x}) > u$ then the same holds in a neighborhood of $\vec{x}$. (Note that essentially h_3 is defined as a supremum.) Hence $h_3(\vec{x}) > u$ if and only if there exists a segment $[\vec{a}, \vec{b}] \subseteq \mathbb{R}^2$ parallel to $\vec{x}$ over which the slope of f is bigger than u. But then by the continuity of f the same holds for segments close enough to $[\vec{a}, \vec{b}]$, in particular to slightly rotated copies, and we are done.

(3.) It is easy to see that for every $\vec{v} \in \mathbb{S}^1$ the slope of f over a segment parallel to $\vec{v}$ is at most the slope of the vector $h(\vec{v})$. Let $\vec{z} \in \mathbb{S}^1$ be an element of the shorter arc connecting $\vec{x}$ and $\vec{y}$ in $\mathbb{S}^1$, let $[\vec{a}, \vec{b}] \subseteq \mathbb{R}^2$ be a segment parallel to $\vec{z}$, and let $P\colon \mathbb{R}^2 \to \mathbb{R}$ be the linear map whose graph passes through the origin, $h(\vec{x})$ and $h(\vec{y})$. (Then graph(P) contains the great circle determined by $h(\vec{x})$ and $h(\vec{y})$.) Moreover, the slope of P over any vector parallel to $\vec{x}$ is the slope of $h(\vec{x})$, and similarly for $\vec{y}$.) We have to show that the slope of f between $\vec{a}$ and $\vec{b}$ is at most that of P, that is, $f(\vec{b}) - f(\vec{a}) \le P(\vec{b}) - P(\vec{a})$. Write $\vec{b} - \vec{a} = \alpha\vec{x} + \beta\vec{y}$ for some $\alpha, \beta > 0$. Then by using the definition of P and our first observation for the segments $[\vec{a}, \vec{a} + \alpha\vec{x}]$ and $[\vec{a} + \alpha\vec{x}, \vec{a} + \alpha\vec{x} + \beta\vec{y}]$, which are parallel to $\vec{x}$ and $\vec{y}$, respectively, we obtain

$$\begin{aligned} f(\vec{b}) - f(\vec{a}) &= f(\vec{a} + \alpha\vec{x} + \beta\vec{y}) - f(\vec{a}) \\ &= (f(\vec{a} + \alpha\vec{x} + \beta\vec{y}) - f(\vec{a} + \alpha\vec{x})) + (f(\vec{a} + \alpha\vec{x}) - f(\vec{a})) \\ &\le (P(\vec{a} + \alpha\vec{x} + \beta\vec{y}) - P(\vec{a} + \alpha\vec{x})) + (P(\vec{a} + \alpha\vec{x}) - P(\vec{a})) \\ &= P(\vec{b}) - P(\vec{a}). \end{aligned}$$

The proof of the lemma is complete. □

4.2.5 Determining the possible S_f's

Definition 4.34. For $c > 0$ let $\psi_c\colon \mathbb{S}^2 \to \mathbb{S}^2$ denote the map that 'deforms S_f according to the map $c \mapsto cf$', that is,

$$\psi_c((x, y, z)) = \frac{(x, y, cz)}{|(x, y, cz)|} \quad \left((x, y, z) \in \mathbb{S}^2\right).$$

Remark 4.35. Let φ_c be the isometry mapping graph(f) onto graph(cf). Every isometry φ is of the form $\varphi^{trans} \circ \varphi^{ort}$, where φ^{ort} is an orthogonal transformation and φ^{trans} is a translation. Moreover, if φ is orientation-preserving then φ^{ort} is a rotation around a line passing through the origin. A key observation is the following: The vertical rigidity of f for C implies that $\psi_c(S_f) = \varphi_c^{ort}(S_f)$ for every $c \in C$.

Now we are ready to prove the following theorem. For the definition of h_3 see the previous subsection.

Theorem 4.36. *Let $C \subseteq (0, \infty)$ be a set condensating to ∞, and let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function vertically rigid for C. Then one of the following holds.*

Case A. *There is a vertical great circle that intersects S_f in only two points.*

Case B. $S_f = \mathbb{S}^2 \setminus \{(0,0,1),(0,0,-1)\}$.

Case C. *There exists an* $\vec{x}_0 \in \mathbb{S}^1$ *such that* $h_3(\vec{x}_0) = 0$ *and* $h_3(\vec{x}) = 1$ *for every* $\vec{x} \neq \vec{x}_0$*, that is,* S_f *is '*$\mathbb{S}^2$ *minus two quarters of a great circle'.*

Case D. *There exists a closed interval* I *in* $\mathbb{S}^1$ *with* $0 < \mathrm{length}(I) < \pi$ *such that* $h_3(\vec{x}) = 0$ *if* $\vec{x} \in I$*, and* $h_3(\vec{x}) = 1$ *if* $\vec{x} \notin I$*, that is,* S_f *is '*$\mathbb{S}^2$ *minus two spherical triangles'.*

Proof. In this proof the word 'component' will refer to the components of $\mathbb{S}^2 \setminus S_f$. We separate two cases according to whether $h_3 \geq 0$ everywhere or not.

First let us suppose that there exists a $\vec{x} \in \mathbb{S}^1$ such that $h_3(\vec{x}) < 0$. This implies that there is a vertical great circle containing two arcs, one in the top component connecting $(0,0,1)$ with $\mathbb{S}^1$ and even crossing it, and an other one (the symmetric pair in the bottom component) running from the 'South Pole to the Equator' and even above. But then considering geometrically the action of ψ_c one can easily check that if we choose larger and larger c's (tending to ∞) then we obtain that $\psi_c(S_f)$ contains in the two components two symmetrical arcs on the same great circle which are only leaving out two small gaps of length tending to 0. But then by Remark 4.35 S_f also contains two such arcs in the two components on some (not necessarily vertical) great circle, hence the distance of the components is 0.

Let $\vec{p}_n$ and $\vec{q}_n$ be sequences in the top and bottom component, respectively, so that $\mathrm{dist}(\vec{p}_n, \vec{q}_n) \to 0$. By compactness we may assume $\vec{p}_n, \vec{q}_n \to \vec{p} \in \mathbb{S}^2$. We claim that $\vec{p}_n \to \vec{p}$ implies $\vec{p} \neq (0,0,-1)$. (And similarly $\vec{q}_n \to \vec{p}$ implies $\vec{p} \neq (0,0,1)$.) Indeed, let $\vec{x}_n \in \mathbb{S}^1$ be so that $\vec{x}_n$ and $\vec{p}_n$ lay on the same vertical great circle, and similarly, let $\vec{x} \in \mathbb{S}^1$ and $\vec{p}$ lay on the same vertical great circle. Then $\vec{x}_n \to \vec{x}$, and using the fact $h(\vec{x}) \neq (0,0,-1)$ and the lower semi-continuity of h at $\vec{x}$ (Lemma 4.33 (1.) and (2.)) we are done.

Using the lower semi-continuity of h at $\vec{x}$ again (and $\vec{p}_n \to \vec{p}$) we obtain that $h(\vec{x})$ cannot be above $\vec{p}$. Similarly, $-h(-\vec{x})$ cannot be below $\vec{p}$. But $h(\vec{x})$ is always above $-h(-\vec{x})$, so the only option is $h(\vec{x}) = -h(-\vec{x})$, hence there is a vertical great circle whose intersection with S_f is just a (symmetric) pair of points, so Case A holds, and hence we are done with the first half of the proof.

Now let us assume that $h_3 \geq 0$ everywhere. First we prove that $h_3(\vec{x}) \in \{0,1\}$ for Lebesgue almost every $\vec{x} \in \mathbb{S}^1$. Indeed, fix an arbitrary $c \in C \setminus \{1\}$. By rigidity the (equal) measure of the two components remains the same after applying ψ_c. Since $h_3 \geq 0$, the intersection of the top component with the vertical great circle containing an $\vec{x}$ shrinks if $c > 1$ and grows if $c < 1$, unless $h_3(\vec{x}) = 0$ or 1. Hence we are done, since the measure of the top component can be calculated from the lengths of these arcs.

Now we show that $\{\vec{x} : h_3(\vec{x}) = 0\}$ is either empty, or a pair of points of the form $\{\vec{x}_0, -\vec{x}_0\}$, or a closed interval in $\mathbb{S}^1$ (possibly degenerate or the whole $\mathbb{S}^1$). So we have to show that if $\vec{x}, \vec{y} \in \mathbb{S}^1$ are so that the shorter arc connecting them is shorter than π, and $h_3(\vec{x}) = h_3(\vec{y}) = 0$ then $h_3(\vec{z}) = 0$ for every $\vec{z}$ in this arc. But $h_3(\vec{z}) \geq 0$ by assumption, and $h_3(\vec{z}) \leq 0$ by the convexity of h applied to $h(\vec{x}) = \vec{x}$ and $h(\vec{y}) = \vec{y}$. The fact that the endpoints are also contained in $\{\vec{x} : h_3(\vec{x}) = 0\}$ easily follows from the semi-continuity.

If $\{\vec{x} : h_3(\vec{x}) = 0\}$ is a symmetrical pair of points or a closed interval of length at least π then it is easy to see that Case A holds. Hence we may assume that it is empty, or a singleton, or a closed interval I with $0 < \text{length}(I) < \pi$.

Case 1. $\{\vec{x} : h_3(\vec{x}) = 0\} = \emptyset$.

In this case, $h_3 > 0$ everywhere, and hence $h_3 = 1$ almost everywhere. Therefore one can easily see (using the convexity) that $h_3 = 1$ everywhere but possibly at at most two points of the form $\{\vec{x}_0, -\vec{x}_0\}$. We claim that actually $h_3 = 1$ everywhere. We know already that S_f is $\mathbb{S}^2$ minus two symmetric arcs on the same vertical great circle. The arcs contain $(0, 0, 1)$ and $(0, 0, -1)$, respectively, and they do not reach the 'Equator', since $h_3 > 0$. Let us fix an arbitrary $c \in C \setminus \{1\}$. By rigidity the (equal) length of the arcs should not change when applying ψ_c, but it clearly changes, a contradiction.

Hence $S_f = \mathbb{S}^2 \setminus \{(0, 0, 1), (0, 0, -1)\}$, so Case B holds.

Case 2. $\{\vec{x} : h_3(\vec{x}) = 0\}$ is a singleton.

Let $\{\vec{x}_0\} = \{\vec{x} : h_3(\vec{x}) = 0\}$. Similarly as above, $h_3 = 1$ almost everywhere. Then convexity easily implies that $h_3(\vec{x}) = 1$ whenever $\vec{x} \notin \{\vec{x}_0, -\vec{x}_0\}$. Again similarly, the length of the arcs is unchanged by ψ_c only if $h_3(-\vec{x}_0) = 1$, so S_f is $\mathbb{S}^2$ minus two symmetric quarter arcs starting from the 'Poles' on a vertical great circle, so Case C holds.

Case 3. $\{\vec{x} : h_3(\vec{x}) = 0\}$ is a closed interval in $\mathbb{S}^1$ with $0 < \text{length}(I) < \pi$.

Let $I = \{\vec{x} : h_3(\vec{x}) = 0\}$. As $h_3 = 0$ or 1 almost everywhere, convexity readily implies that $h_3 = 1$ on $\mathbb{S}^1 \setminus I$. Hence S_f is '$\mathbb{S}^2$ minus two spherical triangles', and Case D holds.

This concludes the proof. □

4.2.6 The end of the proof

Now we complete the proof of Theorem 4.24.

Proof of Theorem 4.24. By Theorem 4.36 it suffices to consider Cases A-D.

Case A. There is a vertical great circle that intersects S_f in only two points.

We may assume using a suitable rotation around the z-axis that the vertical great circle is in the yz-plane, hence $f(x, y)$ is of the form $g(x) + dy$. The continuity of f implies that g is also continuous.

Subcase A1. $d = 0$.

Let $c \in C$ be fixed, and let φ_c be the corresponding isometry. The graph of cf is invariant under translations parallel to the y-axis. As the same holds for f, by rigidity, cf is also invariant under translations parallel to the φ_c-image of the y-axis. If these two directions are nonparallel, then graph(cf) is a plane, and hence so is graph(f), so we are done since $f(x, y)$ is of the form $a + bx$ (note that there is no '$+dy$' since f does not depend on y). Therefore we may assume that all lines parallel to the y-axis are taken to lines parallel to the y-axis, but then all planes parallel to the xz-plane are taken to planes parallel to the xz-plane. But this shows (by considering the intersections of the graphs with the xz-plane) that g is vertically rigid for c, hence by Theorem 4.4 $g(x)$ is of the from $a + bx$ or $a + be^{kx}$ ($a, b, k \in \mathbb{R}$, $k \neq 0$), and we are done.

Subcase A2. $d \neq 0$.

We may assume that $d > 0$, since otherwise we may consider $-f$.

For every $c \in C$ let φ_c be the corresponding isometry. We claim that we may assume that all these are orientation-preserving. If $\{c \in C : \varphi_c \text{ is orientation-preserving}\}$ condensates to ∞ then we are done by shrinking C, otherwise we may assume that they are all orientation-reversing (note that if we split C into two pieces then at least one of them still condensates to ∞). Let us fix a $c_0 \in C$ and consider $c_0 f$ instead of f. By Lemma 4.27 this function is rigid for a set condensating to ∞ with all isometries orientation-preserving, and if it is of the desired form then so is f, so we are done.

We may assume $1 \notin C$. Let us fix a $c \in C$. Similarly as in the previous subcase, we may assume that lines parallel to $(0, 1, d)$ are taken to lines parallel to $(0, 1, cd)$ as follows. The special form of f implies that graph(f) is invariant under translations in the $(0, 1, d)$-direction, hence graph(cf) is invariant under translations in the $(0, 1, cd)$-direction, moreover, by rigidity, graph(cf) is also invariant under translations parallel to the φ_c-image of the lines of direction $(0, 1, d)$. If these two latter directions do not coincide then graph(cf) is a plane, and we are done.

Therefore the image of every line parallel to $(0, 1, d)$ is a line parallel to $(0, 1, cd)$ under the orientation-preserving isometry φ_c. As in Remark 4.35, write $\varphi_c = \varphi_c^{trans} \circ \varphi_c^{ort}$, where φ_c^{ort} is a rotation about a line containing the origin and φ_c^{trans} is a translation. Since the translation does not affect directions, the rotation φ_c^{ort} takes the direction $(0, 1, d)$ to the nonparallel direction $(0, 1, cd)$ ($d \neq 0$), therefore the axis of the rotation has to be orthogonal to the plane spanned by these two directions. Hence the axis has to be the x-axis. Moreover, the angle of the rotation is easily seen to be $\arctan(cd) - \arctan(d)$.

We now show that we may assume that φ_c^{trans} is a horizontal translation. Decompose the translation as $\varphi_c^{trans} = \varphi_c^{\vec{u}} \circ \varphi_c^{\vec{v}}$, where $\varphi_c^{\vec{v}}$ is a horizontal translation and $\varphi_c^{\vec{u}}$ is a translation in the $(0,1,cd)$-direction. Since $\varphi_c^{ort}(\text{graph}(f))$ is invariant under translations in the $(0,1,cd)$-direction, so is $\varphi_c^{\vec{v}} \circ \varphi_c^{ort}(\text{graph}(f))$, hence

$$\begin{aligned}\varphi_c^{\vec{v}} \circ \varphi_c^{ort}(\text{graph}(f)) &= \varphi_c^{\vec{u}} \circ \varphi_c^{\vec{v}} \circ \varphi_c^{ort}(\text{graph}(f)) \\ &= \varphi_c(\text{graph}(f)) = \text{graph}(cf),\end{aligned}$$

so we can assume $\varphi_c = \varphi_c^{\vec{v}} \circ \varphi_c^{ort}$, and we are done.

We will now complete the proof of this subcase by showing that the function $-\frac{1}{d}g$ is rigid for an uncountable set. Indeed, this suffices by Theorem 4.4 and by the special form of f.

Let us denote the xy-plane by $\{z=0\}$ and consider the intersection of both sides of the equation $\varphi_c\left(\text{graph}(f)\right) = \text{graph}(cf)$ with $\{z=0\}$. On the one hand,

$$\begin{aligned}\{z=0\} \cap \varphi_c\left(\text{graph}(f)\right) &= \{z=0\} \cap \varphi_c^{\vec{v}} \circ \varphi_c^{ort}\left(\text{graph}(f)\right) \\ &= \varphi_c^{\vec{v}}\left(\{z=0\} \cap \varphi_c^{ort}\left(\text{graph}(f)\right)\right) \\ &= \varphi_c^{\vec{v}}\left(\text{graph}\left(-w_{c,d}g\right)\right) \\ &= \varphi_c^{\vec{v}}\left(\text{graph}\left((w_{c,d}d)\left(-\frac{1}{d}g\right)\right)\right),\end{aligned}$$

where we used the fact that $\varphi_c^{\vec{v}}$ is horizontal and Lemma 4.25. On the other hand, it is easy to see that

$$\{z=0\} \cap \text{graph}(cf) = \text{graph}\left(-\frac{1}{d}g\right).$$

Therefore $\text{graph}\left(-\frac{1}{d}g\right) = \varphi_c^{\vec{v}}\left(\text{graph}\left((w_{c,d}d)\left(-\frac{1}{d}g\right)\right)\right)$ and hence $-\frac{1}{d}g$ is rigid for $w_{c,d}d$ for every $c > 0$. The map $c \mapsto w_{c,d}d$ is strictly monotone for every fixed d, hence the range of C is uncountable. So $-\frac{1}{d}g$ is rigid for an uncountable set, and we are done.

Case B. $S_f = \mathbb{S}^2 \setminus \{(0,0,1),(0,0,-1)\}$.

So S_f is invariant under every ψ_c, and hence so is under every φ_c^{ort}. Then clearly $\varphi_c^{ort}((0,0,1)) = (0,0,1)$ or $\varphi_c^{ort}((0,0,1)) = (0,0,-1)$ for every $c \in C$. By the same argument as above we can assume that the former holds for every $c \in C$. Using the argument again we can assume that all φ_c's are orientation-preserving. But then each of these is a rotation around the z-axis followed by a translation, in other words, an orientation-preserving transformation in the xy-plane followed by a translation in the z-direction. An orientation-preserving transformation in the plane is either a translation or a rotation. If it is a translation for every c then we are done by Corollary 4.29. So let us assume that there exists a c such that φ_c is a proper rotation around $\vec{x} \in \mathbb{R}^2$ followed by a vertical translation. We claim that then f is constant, which will contradict that S_f is nearly the full sphere, finishing the proof of this case. We will actually show that f is constant on every closed disc $B(\vec{x}, R)$ centered at $\vec{x}$. Indeed, consider $\max_{B(\vec{x},R)} f - \min_{B(\vec{x},R)} f$. This is unchanged by the rotation around $\vec{x}$ as well as by the vertical translation, hence by φ_c. But the map $f \mapsto cf$ multiplies this amount by $c \neq 1$, so the only option is $\max_{B(\vec{x},R)} f - \min_{B(\vec{x},R)} f = 0$, and we are done.

Case C. There exists an $\vec{x}_0 \in \mathbb{S}^1$ such that $h_3(\vec{x}_0) = 0$ and $h_3(\vec{x}) = 1$ for every $\vec{x} \neq \vec{x}_0$, that is, S_f is '$\mathbb{S}^2$ minus two quarters of a great circle'.

So S_f is invariant under every ψ_c, and hence so is under every φ_c^{ort}. Hence φ_c^{ort} maps $(0, 0, 1)$ to one of the four endpoints of the two arcs. Therefore we can assume by splitting C into four pieces according to the image of $(0, 0, 1)$ and applying Lemma 4.27 that $(0, 0, 1)$ is a fixed point of every φ_c^{ort}. But then the two arcs are also fixed, and actually φ_c^{ort} is the identity. Hence every φ_c is a translation, and we are done by Corollary 4.29.

Case D. There exists a closed interval I in $\mathbb{S}^1$ with $0 < \text{length}(I) < \pi$ such that $h_3(\vec{x}) = 0$ if $\vec{x} \in I$ and $h_3(\vec{x}) = 1$ if $\vec{x} \notin I$, that is, S_f is '$\mathbb{S}^2$ minus two spherical triangles'.

As S_f is invariant under every φ_c^{ort}, vertices of the triangles are mapped to vertices. Hence we may assume (by splitting C into six pieces) that $(0, 0, 1)$ is fixed. But then the triangles are also fixed sets, and every φ_c^{ort} is the identity, so we are done as in the previous case.

This finishes the proof of the Theorem 4.24. □

4.3 Horizontally rigid functions

Let us begin with the following definition.

Definition 4.37. A function $f\colon \mathbb{R}^d \to \mathbb{R}$ is *horizontally rigid*, if $\operatorname{graph}(f(c\vec{x}))$ is isometric to $\operatorname{graph}(f(\vec{x}))$ for all $c \in (0, \infty)$.

In this section we characterize the functions of one variable that are horizontally rigid *via translations*. This answers Question 3 of [8] in the case of translations. Note that we do not assume continuity, and analogous theorems hold in every dimension, see [4, Thm. 3.2.].

Theorem 4.38. *A function $f\colon \mathbb{R} \to \mathbb{R}$ is horizontally rigid* via translations *if and only if there exists $r \in \mathbb{R}$ such that f is constant on $(-\infty, r)$ and (r, ∞).*

Proof. These functions are trivially horizontally rigid via translations. As the proof of the other direction resembles that of Theorem 4.7, we only sketch it.

For every $c > 0$ there exist $u_c, v_c \in \mathbb{R}$ such that for all $x \in \mathbb{R}$ we have $f(cx) = f(x + u_c) + v_c$.

We may assume $u_1 = v_1 = 0$. If $c \in (0, \infty) \setminus \{1\}$ then there is an $x_c \in \mathbb{R}$ such that $cx_c = x_c + u_c$, and substituting this back to the above equation we obtain $v_c = 0$. Hence $f(cx) = f(x + u_c)\ \ (x \in \mathbb{R})$ for every $c \in (0, \infty)$.

First we show that if f has a period $p > 0$ then f is constant. Applying the last equation twice we obtain

$$\begin{aligned} f(cx) &= f(x + u_c) = f(x + u_c + p) \\ &= f\left((x + p) + u_c\right) = f\left(c(x + p)\right) \\ &= f(cx + cp). \end{aligned}$$

If x ranges over $\mathbb{R}$ then so does cx, hence cp is also a period. If c ranges over $(0, \infty)$, then so does cp, hence every positive number is a period, so f is constant.

Applying $f(cx) = f(x + u_c)$ again twice we get

$$\begin{aligned} f(c_1(c_2x)) = f(c_2x + u_{c_1}) &= f\left(c_2\left(x + \frac{u_{c_1}}{c_2}\right)\right) \\ &= f\left(x + \frac{u_{c_1}}{c_2} + u_{c_2}\right). \end{aligned}$$

Interchanging c_1 and c_2 and comparing the two equations we obtain

$$f\left(x + \frac{u_{c_1}}{c_2} + u_{c_2}\right) = f\left(x + \frac{u_{c_2}}{c_1} + u_{c_1}\right),$$

so $\pm\left[\left(\frac{u_{c_1}}{c_2} + u_{c_2}\right) - \left(\frac{u_{c_2}}{c_1} + u_{c_1}\right)\right]$ is a period, and hence it is zero. Therefore

$$\frac{u_{c_1}}{1 - \frac{1}{c_1}} = \frac{u_{c_2}}{1 - \frac{1}{c_2}}, \quad c_1, c_2 \in (0, \infty) \setminus \{1\}.$$

Set $r = \frac{u_c}{1 - \frac{1}{c}}$, then $u_c = r\left(1 - \frac{1}{c}\right)$ for every $c \in (0, \infty)$. Substituting this back to $f(cx) = f(x + u_c)$ gives $f(cx) = f\left(x + r\left(1 - \frac{1}{c}\right)\right)$. Writing $\frac{x}{c}$ in place of x yields $f(x) = f\left(\frac{1}{c}(x - r) + r\right)$ for every $c \in (0, \infty)$.

Let $x_0 < r$ be fixed and let c range over $(0, \infty)$, then $\frac{1}{c}(x_0 - r) + r$ ranges over $(-\infty, r)$, so $f(x)$ is constant for $x < r$. Similarly, $f(x)$ is also constant for $x > r$. $\square$

C. Richter showed in [33] that every continuous horizontally rigid function $f\colon \mathbb{R} \to \mathbb{R}$ is of the form $a + bx$. M. Elekes and the author show in [4] that every continuous horizontally rigid function $f\colon \mathbb{R}^2 \to \mathbb{R}$ is of the form $a + bx + dy$. The proof is similar to the proof of Theorem 4.22, the main new ingredient is the use of functional equations.

4.4 Open Questions

Question 4.39. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a vertically (horizontally) rigid function and $c > 0$ such that there exists an isometry between* $\operatorname{graph}(f)$ *and* $\operatorname{graph}(cf)$ *that is not a translation (or also not a reflection). Is then f of the form $a + bx$? Or if we assume some regularity?*

Question 4.40. *Which notion of largeness of C suffices for the various results in dimension two? For example, does the characterization of continuous vertically rigid functions valid if we only assume that C contains three elements that pairwise generate dense multiplicative subgroups of* $(0, \infty)$*?*

Remark 4.41. C. Richter showed in [33] that the continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is vertically rigid for $C \subseteq (0, \infty)$ such that C generates a dense subgroup of $(0, \infty)$ iff f is of the form $a + bx$ or $a + be^{kx}$. However, two independent elements are not enough in the two-dimensional case, since if g is vertically rigid for c_1 via a translation and h is vertically rigid for c_2 via a translation then $f(x, y) = g(x)h(y)$ is vertically rigid for both. Moreover, the main point in that proof in [33] is to replace 'splitting C' by alternative arguments, and we were unable to do so here.

The following question is open in every dimension.

Question 4.42. *Is the same characterization valid for vertically rigid functions if we relax the assumption of continuity to Lebesgue measurability, Baire measurability, Borel measurability, Baire class one or at least one point of continuity?*

Remark 4.43. It would be more natural to replace vertical rigidity by *almost* vertical rigidity. However, it is not clear how this should be defined, as a set can have a measure zero projection on one line and positive measure projection on another.

Question 4.44. *Is there a simple description of rigid sets? Or if we assume some regularity?*

And finally, the most intriguing problem.

Question 4.45. *What are the continuous vertically (horizontally) rigid functions if there are more than two variables?*

Bibliography

[1] R. Balka, *Duality between measure and category in uncountable locally compact abelian Polish groups*, to appear in Real Anal. Exchange.

[2] R. Balka, Z. Buczolich and M. Elekes, *A new fractal dimension: The topological Hausdorff dimension*, submitted.

[3] R. Balka, Z. Buczolich and M. Elekes, *Topological Hausdorff dimension and level sets of generic continuous functions on fractals*, submitted.

[4] R. Balka and M. Elekes, *Continuous horizontally rigid functions of two variables are affine*, submitted.

[5] R. Balka and M. Elekes, *The structure of continuous rigid functions of two variables*, Real Anal. Exchange **35**, no. 1 (2009), 139–156.

[6] R. Balka and M. Elekes, *The structure of rigid functions*, J. Math. Anal. Appl. **345**, no. 2 (2008), 880–888.

[7] T. Bartoszyński, *A note on duality between measure and category*, Proc. Amer. Math. Soc. **128**, no. 9 (2000), 2745–2748.

[8] B. Cain, J. Clark and D. Rose, *Vertically rigid functions*, Real Anal. Exchange **31**, no. 2 (2005/2006), 515–518.

[9] T. J. Carlson, *Strong measure zero and strongly meager sets*, Proc. Amer. Math. Soc. **118**, no. 2 (1993), 577–586.

[10] J. T. Chayes, L. Chayes and R. Durrett, *Connectivity Properties of Mandelbrot's Percolation Process*, Probab. Th. Rel. Fields, **77** (1988), 307–324.

[11] M. Elekes and Á. Tóth, *Covering locally compact groups by less than 2^ω many translates of a compact nullset*, Fund. Math. **197**, (2007), 243–257.

[12] R. Engelking, *Dimension Theory*, North-Holland Publishing Company, 1978.

[13] K. J. Falconer, *The geometry of fractal sets*, Cambridge Tracts in Mathematics No. 85, Cambridge University Press, 1986.

[14] K. Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, Second Edition, John Wiley & Sons, 2003.

[15] K. Falconer and G. Grimmett, *The critical point of fractal percolation in three and more dimensions*, J. Phys. A: Math. Gen., **24** (1991), 491–494.

[16] D. H. Fremlin, *Measure Theory, vol. 4, Topological Measure Spaces*, Torres Fremlin, 2003.

[17] G. Grimmett, *Percolation*, Second Edition, Springer-Verlag, 1999.

[18] P. R. Halmos, *Measure theory*, Springer, 1974.

[19] J. Hawkes, *Trees generated by a simple branching process*, J. London Math. Soc., **24** (1981), 373–384.

[20] E. Hewitt and K. A. Ross, *Abstract harmonic analysis*, Vol. I., Springer, 1963.

[21] G. Hjorth, *Classification and orbit equivalence relations*, American Mathematical Society, Providence, RI, 2000.

[22] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, 1948.

[23] A. S. Kechris, *Classical descriptive set theory*, Springer-Verlag, 1995.

[24] B. Kirchheim, *Hausdorff measure and level sets of typical continuous mappings in Euclidean spaces*, Trans. Amer. Math. Soc., **347** (1995), no. 5, 1763–1777.

[25] T. W. Körner, *Besicovitch via Baire*, Studia Math., **158** (2003), no. 1, 65–78.

[26] M. Kysiak, *Another note on duality between measure and category*, Bull. Pol. Acad. Sci. Math. **51**, no. 3 (2003), 269–281.

[27] B. Mandelbrot, *Les objets fractals. Forme, hasard et dimension.* Nouvelle Bibliothèque Scientifique, Flammarion, Editeur, Paris, 1975.

[28] B. Mandelbrot, *Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier*, J. of Fluid Mechanics, **62** (1974), 331–358.

[29] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Studies in Advanced Mathematics No. 44, Cambridge University Press, 1995.

[30] P. Mörters and Y. Peres, *Brownian motion*, With an appendix by Oded Schramm and Wendelin Werner. *Cambridge University Press*, 2010.

[31] J. von Neumann, *Ein System algebraisch unabhängiger Zahlen*, *Math. Ann.* **99**, (1928), 134–141.

[32] J. C. Oxtoby: *Measure and Category. A survey of the analogies between topological and measure spaces.* Second edition. Graduate Texts in Mathematics No. 2, Springer-Verlag, 1980.

[33] C. Richter, *Continuous rigid functions*, Real Anal. Exchange **35**, no. 2 (2009), 343–354.

[34] W. Rudin, *Fourier analysis on groups*, John Wiley & Sons, 1990.

[35] M. Urbański, *Transfinite Hausdorff dimension*, Topology Appl., **156** (2009), no. 17, 2762–2771.

Summary

In this thesis we study three problems of real analysis. The concepts of measure and category play an important role in each chapter.

In Chapter 2 we introduce a new concept of dimension for metric spaces, the so called *topological Hausdorff dimension*. We examine the basic properties of this new notion of dimension, compare it to other well-known notions, determine its value for some classical fractals such as the Sierpiński carpet, the von Koch snowflake curve, Kakeya sets, trail of the Brownian motion, etc. As our first application, we generalize the celebrated result of Chayes, Chayes and Durrett about the phase transition of the connectedness of the limit set of Mandelbrot's fractal percolation process. As our second application, we show that the topological Hausdorff dimension is precisely the right notion to describe the Hausdorff dimension of the level sets of the generic continuous function (in the sense of Baire category) defined on a compact metric space. This is a joint work with Z. Buczolich and M. Elekes.

In Chapter 3 we study a problem concerning duality between measure and category. Let G be a locally compact abelian (LCA) Polish group. We call a bijection $f\colon G \to G$ an *Erdős–Sierpiński mapping*, if f and f^{-1} map meager sets into Haar null sets and vice versa. The classical Erdős–Sierpiński Theorem states, that assuming CH there exists an Erdős-Sierpiński mapping on $\mathbb{R}$. The question of Ryll-Nardzewski was the following: Is it consistent that there is an addition preserving Erdős–Sierpiński mapping on $\mathbb{R}$? First Bartoszyński gave a negative answer to the question in the case $G = 2^\omega$ [7], then Kysiak solved the original question of Ryll-Nardzewski in [26]. We generalize their results by showing that there is no addition preserving Erdős–Sierpiński mapping on any uncountable LCA Polish group.

Chapter 4 deals with a problem of classical real function theory. Cain, Clark and Rose [8] defined a function $f\colon \mathbb{R}^d \to \mathbb{R}$ to be *vertically rigid* if $\mathrm{graph}(cf)$ is isometric to $\mathrm{graph}(f)$ for all $c > 0$. We prove Janković's conjecture by showing that a continuous function $f\colon \mathbb{R} \to \mathbb{R}$ is vertically rigid if and only if it is of the form $a + bx$ or $a + be^{kx}$ ($a, b, k \in \mathbb{R}$). We answer a question of Cain, Clark and Rose in the negative. We discuss also the Lebesgue (Baire) measurable cases by applying geometric measure (Baire category) theory. We also characterize the continuous vertically rigid functions of two variables. Finally, we obtain results about the so-called horizontally rigid functions. This is a joint work with M. Elekes.

Magyar nyelvű összefoglalás

A disszertáció három valós függvénytani problémát tárgyal, a mérték és a kategória fogalma minden fejezetben fontos szerepet játszik.

A 2. fejezetben egy új dimenzió fogalmat vezetünk be metrikus terek esetén, az úgynevezett *topologikus Hausdorff dimenziót*. Megvizsgáljuk a fogalom tulajdonságait, és összevetjük más, ismert dimenzió fogalmakkal. Meghatározzuk az értékét néhány klasszikus fraktál: a Sierpiński-háromszög, Sierpiński-szőnyeg, von Koch görbe, Kakeya-halmazok és a Brown-mozgás pályája esetén. Első alkalmazásként általánosítjuk Chayes, Chayes és Durrett nevezetes eredményét a Mandelbrot fraktál perkoláció összefüggőségének fázisátmenetéről. Második alkalmazásként megmutatjuk, hogy a topologikus Hausdorff dimenzió pontosan leírja a kompakt metrikus téren értelmezett tipikus folytonos függvény (Baire kategória értelemben) szinthalmazainak Hausdorff-dimenzióját. A fejezet eredményei Buczolich Zoltánnal és Elekes Mártonnal közösek.

A 3. fejezetben a mérték és kategória közötti dualitással foglalkozunk. Legyen G egy lokálisan kompakt kommutatív (LCA) lengyel csoport. Az $f\colon G \to G$ bijekciót *Erdős–Sierpiński leképezésnek* hívjuk, ha f és f^{-1} az első kategóriájú halmazokat Haar-nulla halmazokba képezik, és fordítva. A klasszikus Erdős–Sierpiński Tétel kimondja, hogy CH mellett létezik Erdős–Sierpiński leképezés a számegyenesen. Ryll-Nardzewski a következőt kérdezte: Konzisztens-e, hogy létezik additív Erdős-Sierpiński leképezés a számegyenesen? Először Bartoszyński adott tagadó választ a kérdésre a $G = 2^\omega$ esetben [7], majd Kysiak oldotta meg az eredeti problémát [26]. Az ő eredményeiket általánosítva belátjuk, hogy semmilyen nem megszámlálható LCA lengyel csoporton sem létezik additív Erdős–Sierpiński leképezés.

A 4. fejezet egy klasszikus valós függvénytani problémát tárgyal. Cain, Clark és Rose [8] cikke alapján *függőlegesen merevnek* hívunk egy $f\colon \mathbb{R}^d \to \mathbb{R}$ függvényt, ha $\operatorname{graph}(cf)$ és $\operatorname{graph}(f)$ egybevágóak minden $c > 0$ esetén. Igazoljuk Janković sejtését, azaz megmutatjuk, hogy a folytonos $f\colon \mathbb{R} \to \mathbb{R}$ függvény pontosan akkor függőlegesen merev, ha $a + bx$ vagy $a + be^{kx}$ $(a, b, k \in \mathbb{R})$ alakú. Tagadó választ adunk Cain, Clark és Rose egy kérdésére. Geometriai mértékelméleti módszereket használva vizsgáljuk a Lebesgue (Baire) mérhető függvények esetét. Karakterizáljuk a kétváltozós folytonos függőlegesen merev függvényeket. Végül egy vízszintesen merev függvényekről szóló eredményünk is helyt kap. A fejezet eredményei Elekes Mártonnal közösek.